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Ph.D. Thesis

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A semigroup approach to the space-fractional diffusion and the analysis of fractional Stefan models

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Streszczenie

Głównym celem pracy jest teoretyczna analiza pewnego modelu dyfuzji, która jest nielokalna w przestrzeni. W tym celu zaprezentujemy teorię półgrup analitycznych dla operatora danego w postaci dywergencji z pochodnej Caputo. Następnie, wykorzystamy te rezultaty do rozwiązania jednofazowego, jednowymiarowego ułamkowego w przestrzeni zagadnienia Stefana. Znajdziemy również specjalne rozwiązanie tego problemu metodą rozwiązań samopodobnych. Ostatnia część pracy jest poświęcona ułamkowemu w czasie jednofazowemu, jednowymiarowemu zagadnieniu Stefana. Wyprowadzimy model, zakładając, że strumień dyfuzji dany jest w postaci ułamkowej względem czasu pochodnej Riemanna-Liouville z gradientu gęstości transportowanej substancji. Znajdziemy też specjalne rozwiązanie tego problemu.

Słowa kluczowe: teoria półgrup analitycznych, ułamkowe zagadnienia Stefana, rozwiązania samopodobne

Abstract

This work mainly concentrates on providing the mathematical background for a specific model of fractional in space diffusion. We will develop the theory of analytic semigroups for an operator given by divergence of fractional Caputo derivative. Subsequently, we will apply these results to obtain a solution to one-phase, one-dimensional fractional in space Stefan problem. We will also find a special solution to this problem by similarity variable method. The final part of thesis is devoted to fractional in time one-phase, one-dimensional Stefan model. We derive a model assuming that the diffusive flux is given by the time-fractional Riemann-Liouville derivative of gradient of transported substance. Then, we will obtain a special solution to this problem.

Key words: analytic semigroup theory, fractional Stefan problems, self-similar solutions

Contents

Chapter 1

Motivation

In this work we study problems exhibiting non-local in space or non-local in time effects. The non-Fickian diffusion has been already observed in complex, heterogeneous media. An overview of recently derived models may be found in [5]. A model phenomenon considered in [5] is a mass transport in fractured porous aquifer. In such complex domain we expect different behaviour of fluid in fractures and different in pores. Hence, we may regard each of phases (fluid in porous blocks and fluid in network of fractures) as a continuum that occupies the entire domain. Moreover, we take into account the mass exchange between this two continua. This idea, called double-continua approach, was proposed in [3] and it leads to a model that has a non-local character. We refer to [5, chapter 1.2.3] for a detailed derivation of this model. Let us discuss the case when the length scales of heterogeneity of medium are assumed to be power-low distributed. Such situation is quite extensively discussed in recent literature. We refer to [5, chapter 1.2.5] and references therein, for the case of fractured porous medium. The main idea of modelling diffusion processes in such media is to assume that the diffusive flux is proportional to the fractional derivative of transported quantity.

In this work, we mainly focus on an anomalous super-diffusion model, where the diffusive flux is given by the fractional Caputo derivative with respect to space variable. Such an idea was introduced by V. Voller in [31], where the author considered the model of infiltration of water into heterogeneous soils. Subsequently, the author transferred the idea of representing the diffusive operator as a divergence of Caputo derivative, to the one-phase Stefan problem (see [32]). One of the goals of this paper is to investigate the mathematical properties of this operator from the operator theory perspective. The results concerning this issue are presented in Chapter 3. In this chapter we solve the super-diffusion problem with various kinds of boundary conditions by means of an analytic semigroup theory. We note that most of the results of the first section of Chapter 3 come from [27]. We emphasise that we develop the theory of analytic semigroups in L^2 - framework. In the final section of this chapter we present the approach to solve the super-diffusion problem in the case where data do not belong to L^2. This method provides us weak solutions by means of energy estimates.

In Chapter 4 we present an application of the results of previous chapter. We solve the one-phase, one-dimensional, space-fractional Stefan model introduced in [32]. In the proof we apply the theory of evolution operators based on the results obtained in the first section of Chapter 3. Subsequently, we increase the regularity of obtained solution in the interior of the domain. In the second section of Chapter 4, we derive space-fractional versions of maximum principles and Hopf lemma. Finally, we apply the Schauder fixed point theorem, to obtain the solution to Stefan problem. We note that the results of Chapter 4, described above, come from [28]. We finish this chapter with an example of an exact solution to space-fractional Stefan problem by means of similarity variable method.

In the final part of the thesis we concentrate on anomalous sub-diffusion model with temporal non-locality. Here, we are motivated by [33], where the authors represent the non-locality in time, assuming that the diffusive flux is given in the form of the time-fractional Riemann-Liouville derivative of temperature gradient. In a final part of introductory Chapter 2, we present a careful derivation of one-phase, one-dimensional Stefan problem based on such assumption on the flux. The existence of special, self-similar solution to this problem will be proven in Chapter 5. Furthermore, we will show a uniform convergence of self-similar solutions to the time-fractional Stefan problem to a self-similar solution to the classical Stefan problem as a fractional parameter α tends to one. The results concerning sub-diffusion effects come from [14].

Chapter 2

Introduction

The introductory chapter is divided into the two parts. The first section summarizes without proofs the relevant material from mathematical analysis, the theory of operators as well as the theory of semigroups and evolution operators. In the subsequent sections we turn our attention to fractional calculus. The sections second and third are devoted to the preliminary results concerning fractional operators, considered in this work. We introduce their notions and we give a brief exposition of their properties. We finish this chapter with derivation of two fractional Stefan models which will be considered in subsequent chapters.

2.1. Preliminaries

2.1.1. Function spaces

The absolutely continuous functions play an essential role in the theory of fractional calculus. Here, we recall their definition and characterization.

Definition 2.1. *[16, Definition 3.1] If $P \subseteq \mathbb{R}$, then we say that $f : P \to \mathbb{R}$ is absolutely continuous on P ($f \in AC(P)$) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every finite number of non overlapping intervals (a_k, b_k), $k \in 1, \ldots, n$ with $[a_k, b_k] \subseteq P$ and

$$\sum_{k=1}^{n} |b_k - a_k| < \delta.$$

We write $f \in AC_{loc}(P)$, if $f \in AC([a, b])$ for every $[a, b] \subseteq P$.

Theorem 2.1. *[16, Theorem 3.30] Let $P \subset \mathbb{R}$ be an interval. A function $f : P \to \mathbb{R}$ belongs to $AC_{loc}(P)$ if and only if*

(i) f is continuous in P,

(ii) f is differentiable a.e. in P and f' belongs to $L^1_{loc}(P)$,

(iii) for all $x, y \in P$

$$f(y) = f(x) + \int_x^y f'(t)dt. \tag{2.1}$$

We note that from the above theorem follows that the space $AC_{loc}(\mathbb{R})$ may be identified with $W^{1,1}_{loc}(\mathbb{R})$.
Let us now introduce the definition of a fractional Sobolev space. The definition will be given by means of complex interpolation. For an introduction to real and complex interpolation we refer to [20].

Definition 2.2. *[17, section 9.1] Let $L > 0$, $\beta > 0$. We choose a natural number m greater than or equal to β. Then, we define a fractional Sobolev space as a complex interpolation space*

$$H^\beta(0, L) := [L^2(0, L), H^m(0, L)]_{\frac{\beta}{m}}.$$

This definition is independent of the choice of number m up to the norm equivalence.

Remark 2.1. *[30, Remark 4.4.2/2] The space $H^\beta(0, L)$ coincides with the space of functions belonging to $L^2(0, L)$ such that for $s = \beta - \lfloor \beta \rfloor$*

$$\sum_{j \leq \lfloor \beta \rfloor} \int_0^L \int_0^L \frac{\left| f^{(j)}(x) - f^{(j)}(y) \right|^2}{|x - y|^{1+2s}} dydx < \infty.$$

The equivalent norm in $H^\beta(0, L)$ is given by

$$\|f\|_{H^\beta(0,L)} = \left(\|f\|^2_{L^2(0,L)} + \sum_{j \leq \lfloor \beta \rfloor} \int_0^L \int_0^L \frac{\left| f^{(j)}(x) - f^{(j)}(y) \right|^2}{|x - y|^{1+2s}} dydx \right)^{\frac{1}{2}}.$$

We will frequently make use of the following remark from [17]. Here we consider only the one dimensional case.

Remark 2.2. *[17, Remark 12.8.] For $L > 0, s \geq 0, s \neq \frac{1}{2}$ there holds*

$$\frac{\partial}{\partial x} \in B(H^s(0, L); H^{s-1}(0, L)).$$

2.1.2. Fractional powers of operators

Here, we present a brief introduction to the theory of fractional powers of operators. We limit ourselves only to the most essential results that will be used in the thesis. Although there are a few standard approaches to this topic, here we follow the one introduced in [21]. It will provide us a uniformity of notation. For a comprehensive study of the fractional powers of operators we refer to [2], [19], [21] [23], [30], [34].
In the whole subsection we discuss only linear operators $A : D(A) \subseteq X \to X$ where X is a Banach space. Here and henceforth by E we denote the identity operator.

Definition 2.3. *[21, Definition 1.1.1] We say that A is non-negative if $(-\infty, 0) \subseteq \rho(A)$ and there exists $M > 0$ such that*
$$\left\|(\lambda E + A)^{-1}\right\|_{B(X)} \le \frac{M}{\lambda} \text{ for every } \lambda > 0.$$

Definition 2.4. *[21, Definition 1.1.2] If A is non-negative and additionally $0 \in \rho(A)$ we say that A is positive.*

We note that if A is injective but it is not invertible, by A^{-1} we understand the operator with the domain $D(A^{-1}) = R(A)$ defined as follows: for every $x \in D(A^{-1})$ we set $A^{-1}x = y$, where $Ay = x$.

Proposition 2.2. *[21, Proposition 1.1.2] If A is non-negative and injective, then A^{-1} is also non-negative.*

Let us pass to the definition of Balakrishnan operator.

Definition 2.5. *[21, Definition 3.1.1] Let A be a non-negative operator. We define for $0 < \operatorname{Re}\alpha < 1$ operator J^α as follows $D(J^\alpha) = D(A)$*
$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + A)^{-1} A u d\lambda.$$
For $n < \operatorname{Re}\alpha < n+1$, we set $D(J^\alpha) = D(A^{n+1})$ and $J^\alpha = J^{\alpha-n}A^n$.
If $\operatorname{Re}\alpha = 1$, $D(J^\alpha) = D(A^2)$ and
$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}[(\lambda + A)^{-1} - \frac{\lambda}{\lambda^2 - 1}] A u d\lambda + \sin\frac{\alpha\pi}{2} A u.$$
For $\operatorname{Re}\alpha = n+1$, we set $D(J^\alpha) = D(A^{n+2})$ and $J^\alpha = J^{\alpha-n}A^n$.

Proposition 2.3. *[21, Theorem 3.1.5 and Theorem 3.1.6] Let A be a non-negative operator. If $u \in D(A^n)$ then $\alpha \mapsto J^\alpha u$ is analytic in $\{\alpha \in \mathbb{C} : 0 < \operatorname{Re}\alpha < n\}$ with values in X. Let A be densely defined. Then for $u \in D(A)$ we have*

1. *$\lim_{\alpha\to 1} J^\alpha u = Au$, where the convergence is in a fixed region contained in $\{\alpha \in \mathbb{C} : 0 < \operatorname{Re}\alpha < 1\}$,*
2. *$\lim_{\alpha\to 0} J^\alpha u = u$, where the convergence is in a fixed region contained in $\{\alpha \in \mathbb{C} : 0 < \operatorname{Re}\alpha < 1\}$.*

The definition of a positive power (i.e. $\operatorname{Re}\alpha > 0$) for non-negative and bounded operator is given by the Balakrishnan operator.

Definition 2.6. *[21, Definition 5.1.1] If A is non-negative and bounded we define $A^\alpha = J^\alpha$ for $\operatorname{Re}\alpha > 0$.*

Definition 2.7. *[21, Definition 5.1.2] Let A be an unbounded and positive operator. We define for $\operatorname{Re}\alpha > 0$*
$$A^\alpha = ((A^{-1})^\alpha)^{-1}.$$
Here, the domain of A^α consists of $u \in X$ such that $u \in R((A^{-1})^\alpha)$.

We also present the definition of A^α in the case where A is unbounded and not invertible. However, we do not concentrate on this topic, because in this work we will discuss the operators which are either bounded or invertible.

Definition 2.8. *[21, Definition 5.1.3] If A is non-negative, unbounded and $0 \in \sigma(A)$, for $\operatorname{Re}\alpha > 0$ and for $u \in X$ such that $u \in D((A+\varepsilon)^\alpha)$ for $\varepsilon > 0$ close to zero and $\lim_{\varepsilon\to 0^+}(A+\varepsilon)^\alpha u$ exists in X, we define*

$$A^\alpha u := \lim_{\varepsilon\to 0^+}(A+\varepsilon)^\alpha u.$$

In the next proposition we present an interpolation estimate between the norms of fractional powers of non-negative operator.

Proposition 2.4. *[34, Remark 2.9, Chapter 2.7.4] Let $0 \le \alpha < \beta < \gamma \le 1$. We assume that A is a non-negative operator in the sense of Definition 2.3. Then for any $u \in D(A^\gamma)$ we have*

$$\left\|A^\beta u\right\| \le \frac{4}{\pi}(M+1)\left\|A^\gamma u\right\|^{\frac{\beta-\alpha}{\gamma-\alpha}}\left\|A^\alpha u\right\|^{\frac{\gamma-\beta}{\gamma-\alpha}},$$

where the constant M comes from Definition 2.3.

Proposition 2.5. *[21, Theorem 3.1.8 and Corollary 5.1.12] Let $\operatorname{Re}\alpha > 0$ and A be an non-negative operator. Then, J^α is closable and $A^\alpha = \overline{J^\alpha}$ if and only if A is densely defined.*

Corollary 2.6. *[21, Corollary 5.2.4] If A is non-negative and injective, then for $\operatorname{Re}\alpha > 0$ A^α is also injective and $(A^{-1})^\alpha = (A^\alpha)^{-1}$.*

Now we introduce negative and imaginary powers.

Definition 2.9. *[21, Definition 7.1.1 and Definition 7.1.2] Let A be non-negative and injective. Then for $\operatorname{Re}\alpha > 0$ we set $A^{-\alpha} := (A^\alpha)^{-1}$. Moreover,*

$$A^{i\tau} := (A+\lambda)^2 A^{-1} A^{1+i\tau}(A+\lambda)^{-2} \quad \text{for} \quad \lambda > 0.$$

If A is invertible, then in the definition of $A^{i\tau}$, we may take $\lambda = 0$.

We finish this subsection with two theorems. We note that the second one is of fundamental importance, if we consider the domains of fractional powers of operators.

Theorem 2.7. *[21, Theorem 7.1.1] Let $\alpha, \beta \in \mathbb{C}$ and let A be a non-negative and injective operator. If $u \in D(A^{\alpha+\beta}) \cap D(A^\beta)$, then $A^\beta u \in D(A^\alpha)$ and $A^\alpha A^\beta u = A^{\alpha+\beta}u$.*

Theorem 2.8. *[21, Theorem 11.6.1] Let us assume that A is densely defined, non-negative operator on a Banach space X. If A has bounded imaginary powers, i.e. A is injective and there exist $M > 0$, $\theta \ge 0$ such that*

$$\left\|A^{it}\right\|_{B(X)} \le M e^{\theta|t|} \quad \text{for every} \quad t \in \mathbb{R},$$

then $D(A^\alpha) = [X, D(A)]_\alpha$ for $\alpha \in (0,1)$ with norm equivalence.

2.1.3. Semigroup theory

The third chapter of the thesis is devoted to an analysis of the operator of space-fractional diffusion from the perspective of semigroup theory. Here, we present a brief introduction to this subject. Let us recall the definitions of C_0 - semigroup of contractions and the infinitesimal generator of the semigroup.

Definition 2.10. *[23, Chapter 1] Let X be a Banach space. Let $T(t)$, $0 \leq t < \infty$ be a one parameter family such that $T(t) \in B(X)$ for every $t \in [0, \infty)$. Then $T(t)$ is called a C_0 - semigroup of contractions iff*

1. $T(0) = E$,
2. $T(t+s) = T(t)T(s)$ *for every* $t, s \in [0, \infty)$,
3. $\lim_{t \to 0^+} T(t)u = u$ *for every* $u \in X$,
4. $\|T(t)\|_{B(X)} \leq 1$.

We note that if the family $T(t)$ satisfies only the first two assumptions it is called a semigroup and if it satisfies additionally 3., then it is called a C_0 - semigroup.

Definition 2.11. *[23, Chapter 1] Let X be a Banach space and let $T(t)$ be a semigroup on X. The linear operator A defined by*

$$D(A) = \{u \in X : \lim_{t \to 0^+} \frac{T(t)u - u}{t} \text{ exists in } X\}$$

and

$$Au : D(A) \to X, \quad Au = \lim_{t \to 0^+} \frac{T(t)u - u}{t}$$

is called the infinitesimal generator of semigroup $T(t)$.

Remark 2.3. *[23, Chapter 1, Theorem 2.4] One of fundamental properties of the C_0 - semigroups is that if $T(t)$ is a C_0 - semigroup and A denotes its generator, then if $u \in D(A)$, then $T(t)u \in D(A)$ and*

$$\frac{d}{dt} T(t)u = AT(t)u = T(t)Au.$$

Let us recall the definition of dissipative operator.

Definition 2.12. *[2, Definition 3.4.1] A linear operator $A : D(A) \subseteq X \to X$ is called dissipative if for every $u \in D(A)$ there exists $u^* \in X^*$ such that $\|u^*\| \leq 1$, $\langle u, u^* \rangle = \|u\|$ and $\mathrm{Re}\langle Au, u^* \rangle \leq 0$.*

We present also a characterization of dissipative operator.

Remark 2.4. *[2, Lemma 3.4.2, Example 3.4.3]*

1. *An operator A on a Banach space X is dissipative if and only if for every $u \in D(A)$ and every $t > 0$ the holds $\|u - tAu\| \geq \|u\|$.*

2. *From the first part of the remark it follows easily that if X is a Hilbert space, A is dissipative if and only if for every $u \in D(A)$ the holds* $\operatorname{Re}(Au, u) \leq 0$.

Let us present the Lumer-Philips theorem which provides the criterion for an operator to be a generator of C_0-semigroup of contractions.

Theorem 2.9. *[2, Theorem 3.4.5] Let X be a Banach space and A be a linear, densely defined operator on X. Then A is a generator of C_0-semigroup of contractions if and only if*

1. *A is dissipative,*
2. *there exists $\lambda > 0$ such that $R(\lambda E - A) = X$.*

We introduce a definition of numerical range and a classical result from [23].

Proposition 2.10. *[23, Ch.1, Theorem 3.9.] Let X be a Banach space. For a linear operator A in X we define its numerical range $S(A)$ as*

$$S(A) = \{\langle x^*, Ax\rangle : x \in D(A), \|x\| = 1, x^* \in X^*, \|x^*\| = 1, \langle x^*, x\rangle = 1\}.$$

Let us assume that A is closed, linear and densely defined on X. We denote by $\Sigma := \mathbb{C}\backslash\overline{S(A)}$. If $\lambda \in \Sigma$, then $\lambda E - A$ is injective and has closed range. Moreover, if $\Sigma_0 \subseteq \Sigma$ is such that $\Sigma_0 \cap \rho(A) \neq \emptyset$, then the spectrum of A is contained in $\mathbb{C} \setminus \Sigma_0$ and

$$\left\|(\lambda E - A)^{-1}\right\| \leq \frac{1}{d(\lambda, \overline{S(A)})} \text{ for every } \lambda \in \Sigma_0,$$

where $d(\lambda, \overline{S(A)})$ denotes a distance between λ and $\overline{S(A)}$.

Now, we will recall the definition of an analytic semigroup. At first, we introduce the definition of a sectorial operator given by [19].

Definition 2.13. *[19, Definition 2.0.1] Let A be a linear operator on a Banach space X. We say that A is sectorial if there exists $M > 0$ and $\omega \in (\frac{\pi}{2}, \pi]$ such that*

1. *$\rho(A) \supseteq S_\omega := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg \lambda| < \omega\}$,*
2. *$\|(\lambda E - A)^{-1}\| \leq \frac{M}{|\lambda|}$ for every $\lambda \in S_\omega$.*

The definition of an analytic semigroup is given as follows.

Definition 2.14. *[19, Definition 2.0.1] Let $A : D(A) \subset X \to X$ be a sectorial operator and let ω be the constant from Definition 2.13. The family $T(t)$ defined by $T(0) = E$ and*

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{t\lambda}(\lambda E - A)^{-1} d\lambda \quad \text{for} \quad t > 0,$$

where $r > 0, \eta \in (\frac{\pi}{2}, \omega)$ and

$$\Gamma_{r,\eta} := \{\lambda \in \mathbb{C} : |\arg \lambda| = \eta,\ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \eta,\ |\lambda| = r\}$$

is the curve oriented counterclockwise, is called an analytic semigroup generated by A.

If a sectorial operator A is densely defined then an analytic semigroup generated by A is in particular a C_0 semigroup.

In fact, the semigroup generated by a densely defined sectorial operator has better regularity properties then a C_0 - semigroup, i.e. it increases the regularity of initial condition. We present here selected properties of analytic semigroups. The following results comes from [34, Theorem 3.4 Chapter 3.2.1] and [19, Proposition 2.1.1].

Theorem 2.11. *Let A be a sectorial operator on a Banach space X and $u_0 \in X$. Then, there exists exactly one solution to*

$$\frac{d}{dt}U(t) = AU(t), \quad U(0) = u_0$$

belonging to $C([0,T];X) \cap C((0,T];D(A)) \cap C^1((0,T];X)$. The solution is given by $U(t) = T(t)u_0$, where $T(t)$ denotes an analytic semigroup generated by A. Furthermore, there exists a positive constant $c = c(T)$, which is an increasing function of T, such that the following estimate holds for every $t \in (0,T]$

$$\|U(t)\|_X + t\|U'(t)\|_X + t\|AU(t)\|_X \le c\|u_0\|.$$

Nevertheless, for every $u \in X$ and every $k \in \mathbb{N}$ there holds $T(t)u \in D(A^k)$ for $t > 0$ and if $u \in D(A^k)$, then $A^kT(t)u = T(t)A^ku$ for every $t \ge 0$. Besides, $T(t)u \in C^\infty((0,\infty);X)$ and

$$\frac{d^k}{dt^k}T(t) = A^kT(t).$$

One may also consider a nonhomogeneous problem. Here we present a simplified version of [34, Theorem 3.4 Chapter 3.2.1]. We note that in our formulation, the regularity of the source term is not optimal. For optimal regularity results we refer to [34] and [19].

Theorem 2.12. *Let A be a sectorial operator on a Banach space X, $u_0 \in X$ and $F \in C^{0,\nu}([0,T];X)$ for $\nu \in (0,1)$. Then, there exists exactly one solution to*

$$\frac{d}{dt}U(t) = AU(t) + F(t), \quad U(0) = u_0 \tag{2.2}$$

in $C([0,T];X) \cap C((0,T];D(A)) \cap C^1((0,T];X)$ which is given by the variation of constant formula

$$U(t) = T(t)u_0 + \int_0^t T(t-\tau)F(\tau)d\tau,$$

where $T(t)$ denotes an analytic semigroup generated by A.

We will also make use of the version of the estimate in complex interpolation spaces. We present here, scaled version of [19, Proposition 2.2.9.]. We note that the original result from [19] is more general.

Proposition 2.13. *[19, Proposition 2.2.9.] (scaled version) Let $T(t)$ be an analytic semigroup generated by sectorial operator A. Then, for every $t \in (0,T)$, $n \in \mathbb{N}$, $\alpha, \beta \in [0,1]$ there holds*

$$\|A^nT(t)u\|_{[X,D(A)]_\beta} \le ct^{\alpha-\beta-n}\|u\|_{[X,D(A)]_\alpha},$$

where c is a positive constant dependent on α, β, n and T. Moreover, c is an increasing function of T.

We finish this subsection with a useful result concerning the perturbation of the generator of an analytic semigroup.

Proposition 2.14. *[19, Proposition 2.4.1] Let X be a Banach space and $A : D(A) \subseteq X \to X$ be sectorial. Let us consider $B \in B(Y, X)$, where Y is a Banach space such that $D(A) \subseteq Y \subseteq X$. We equip $D(A)$ with the graph norm, i.e. $\|u\|_{D(A)} = \|u\|_X + \|Au\|_X$. If there exists $\alpha \in (0, 1)$ and $c > 0$ such that*

$$\|u\|_Y \leq c \|u\|_{D(A)}^{\alpha} \|u\|_X^{1-\alpha} \text{ for every } u \in D(A).$$

Then, $A + B : D(A) \to X$ is sectorial.

2.1.4. Evolution operators

Now we will present a brief introduction to the theory of non-autonomous equations. By means of this theory we solve the space-fractional Stefan problem in Chapter 4. Let as begin with the definition of evolution operator.

Definition 2.15. *[19, Definition 6.0.1] Let X be a Banach space, $T > 0$. A family of linear bounded operators $\{G(t, \sigma) : 0 \leq \sigma \leq t \leq T\}$ is said to be an evolution operator for the problem*

$$u'(t) = A(t)u + f(t), \;\; 0 < t \leq T, \;\; u(0) = u_0,$$

where $A(\cdot)$ denotes a family of sectorial operators with common domains, i.e. $D(A(t)) \equiv D$ for every $t \in [0, T]$, if

1. *$G(t, \sigma)G(\sigma, r) = G(t, r)$, $G(\sigma, \sigma) = E$, for $0 \leq r \leq \sigma \leq t \leq T$,*
2. *$G(t, \sigma) \in B(X, D)$ for $0 \leq \sigma \leq t \leq T$,*
3. *$t \mapsto G(t, \sigma)$ is differentiable in (σ, T) with values in $B(X)$ and*
$$\frac{\partial}{\partial t} G(t, \sigma) = A(t)G(t, \sigma) \text{ for } 0 \leq \sigma < t \leq T.$$

Theorem 2.15. *[19, Chapter 6] Let D be a Banach space continuously embedded into X and let $T > 0$, $a \in (0, 1)$. If for $0 \leq t \leq T$ $A(t) : D(A(t)) \to X$ satisfies*

1. *for every $t \in [0, T]$ $A(t)$ is sectorial and $D(A(t)) \equiv D$,*
2. *$t \mapsto A(t) \in C^{0,a}([0, T]; B(D, X))$,*

then there exists a family of evolution operators for $A(t)$ given by Definition 2.15.

If the initial data are more regular, we expect higher-regularity results up to the initial time.

Proposition 2.16. *[19, Corollary 6.1.6.(i), (iii)] Let $A(t)$ satisfies the assumptions of Theorem 2.15. If $u_0 \in X$, then $G(t,0)u_0 \in C([0,T];X) \cap C^1((0,T];X) \cap C((0,T];D)$. Furthermore, if $u_0 \in D$, then $G(t,0)u_0 \in C^1([0,T];X) \cap C([0,T];D)$ and*

$$\frac{\partial}{\partial t} G(t,0)u_0 = A(t)G(t,0)u_0 \text{ for every } 0 \leq t \leq T.$$

In order to develop the theory of non-homogenous problems we introduce the notion of a mild solution.

Definition 2.16. *Let us discuss the problem*

$$u'(t) = A(t)u(t) + f(t), \;\; \sigma < t \leq T, \;\; u(\sigma) = u_\sigma, \tag{2.3}$$

where $A(t)$ satisfies the assumptions of Theorem 2.15. We denote by $\{G(t,\tau) : \sigma \leq \tau \leq t \leq T\}$ the family of evolution operators generated by $A(t)$. For every $f \in L^1(\sigma,T;X)$, $u_\sigma \in X$ function u defined by the formula

$$u(t) = G(t,\sigma)u_\sigma + \int_\sigma^t G(t,\tau)f(\tau)d\tau \tag{2.4}$$

is called a mild solution to (2.3).

The next proposition establishes when the solution to (2.3) is given by (2.4).

Proposition 2.17. *[19, Corollary 6.2.4.] Let $f \in C((\sigma,T];X) \cap L^1(\sigma,T;X)$, $u_\sigma \in \overline{D}$. If problem (2.3) has a solution belonging to $C^1((0,T];X) \cap C((0,T];D) \cap C([0,T];X)$ so that (2.3) is satisfied for each $t \in (0,T]$, then u is given by (2.4).*

We finish this section with a proposition that collects the estimates which are used in the proof of Theorem 4.1.

Proposition 2.18. *[19, Corollary 6.1.8] Let $\{G(t,\sigma) : 0 \leq \sigma \leq t \leq T\}$ be a family of evolution operators generated by $A(t) : D \to X$. Then for every $\theta, \delta \in (0,1)$, G satisfies the following estimates. If $g \in [X,D]_\delta$, then for any $0 \leq \sigma < t \leq T$ there exists positive constant $c = c(\theta,\delta,T)$ which is a continuous increasing function of T such that*

$$\|G(t,\sigma)g\|_D \leq \frac{c}{(t-\sigma)^{1-\delta}} \|g\|_{[X,D]_\delta}. \tag{2.5}$$

Moreover, for any $0 \leq \delta < \theta < 1$, we have

$$\|G(t,\sigma)g\|_{[X,D]_\theta} \leq \frac{c}{(t-\sigma)^{\theta-\delta}} \|g\|_{[X,D]_\delta} \tag{2.6}$$

and for $\theta \in (0,1)$, $\delta \in (0,1]$

$$\|A(t)G(t,\sigma)g\|_{[X,D]_\theta} \leq \frac{c}{(t-\sigma)^{1+\theta-\delta}} \|g\|_{[X,D]_\delta}. \tag{2.7}$$

Finally, for every $0 \leq \sigma < r < t \leq T$

$$\|A(t)G(t,\sigma)g - A(r)G(r,\sigma)g\|_X \leq c\left(\frac{(t-r)^a}{(r-\sigma)^{1-\delta}} + \frac{1}{(r-\sigma)^{1-\delta}} - \frac{1}{(t-\sigma)^{1-\delta}}\right) \|g\|_{[X,D]_\delta}, \tag{2.8}$$

where $a \in (0,1)$ comes from Theorem 2.15.

2.2. Fractional integrals and derivatives and their connection with fractional powers of operators

According to [29], the origins of fractional derivatives are dated to XVII century and the origins of calculus itself. Nevertheless, this subject became extensively studied just in the last decades. In this section we introduce the definitions of fractional operators and we briefly establish their connection with fractional powers of operators of integration and differentiation.

Definition 2.17. *Let $L > 0$, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$. For $f \in L^1(0,L)$ we introduce the fractional integral I^α by the formula*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-p)^{\alpha-1} f(p)dp. \tag{2.9}$$

Here $\Gamma(\cdot)$ denotes the Gamma function which is given by the formula

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Here and in the whole thesis by $*$ we denote the convolution on $(0,\infty)$, i.e.

$$(f * g)(x) = \int_0^x f(p)g(x-p)dp.$$

We note that the fractional integral is given by convolution on positive real line with integrable kernel, i. e. $I^\alpha f = \frac{x^{\alpha-1}}{\Gamma(\alpha)} * f$. Hence, by the Young inequality for convolution it is well defined for integrable functions and $I^\alpha \in B(L^p(0,L), L^p(0,L))$ for every $p \in [1,\infty]$. Directly from the formula we may notice that $(I^1 f)(x) = \int_0^x f(p)dp$. To show that $I^n f$ is equal to $n-$ fold integral it is enough to apply the integration by parts formula.
Using the phrase 'fractional integral', it seems natural to ask, whether the operator defined by (2.9) may be interpreted as a fractional power of the operator of integration. We will give an affirmative answer to this question, in the case of L^p space for $p \in [1,\infty]$. It is also worth to mention the paper [9], were the case of L^2 - space was considered. Let us define the operator of integration on $L^p(0,L)$ by

$$(If)(x) = \int_0^x f(p)dp \quad \text{for } f \in L^p(0,L), \;\; p \in [1,\infty]. \tag{2.10}$$

We will show that the operator I is non-negative in the sense of Definition 2.3. The non-negativity of I in $L^p(0,L)$ for $p \in [1,\infty]$ follows from the two propositions presented below.

Proposition 2.19. *[12] Let $L > 0$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. If $u, v \in L^1(0,L)$, then*

$$(\lambda E + I)v(t) = u(t) \iff v(t) = \lambda^{-1}u(t) - \lambda^{-2}\int_0^t u(s)e^{\frac{s-t}{\lambda}}ds. \tag{2.11}$$

Proof. Let us assume that $v(t) = \lambda^{-1}u(t) - \lambda^{-2}\int_0^t u(s)e^{\frac{s-t}{\lambda}}ds$. Applying the Fubini theorem we arrive at

$$(\lambda E + I)v(t) = u(t) - \lambda^{-1}\int_0^t u(s)e^{\frac{s-t}{\lambda}}ds + \lambda^{-1}\int_0^t u(s)ds - \lambda^{-2}\int_0^t\int_0^\tau u(s)e^{\frac{s-\tau}{\lambda}}dsd\tau$$

$$= u(t) - \lambda^{-1}\int_0^t u(s)(e^{\frac{s-t}{\lambda}} - 1)ds - \lambda^{-2}\int_0^t\int_s^t e^{\frac{s-\tau}{\lambda}}d\tau u(s)ds = u(t).$$

To obtain the reverse implication we convolve both sides of $(\lambda E + I)v(t) = u(t)$ with $e^{-\frac{t}{\lambda}}$ and we get

$$\lambda\int_0^t v(s)e^{\frac{s-t}{\lambda}}ds + \int_0^t\int_0^\tau v(s)dse^{\frac{\tau-t}{\lambda}}d\tau = \int_0^t u(s)e^{\frac{s-t}{\lambda}}ds.$$

If we apply the Fubini theorem, then the above equality reduces to

$$\lambda\int_0^t v(s)ds = \int_0^t u(s)e^{\frac{s-t}{\lambda}}ds.$$

Taking the derivative of both sides we arrive at the desired equality. □

Proposition 2.20. *[12] If $L > 0$, $p \in [1,\infty]$, $\lambda \neq 0$, then $I + \lambda E : L^p(0,L) \longrightarrow L^p(0,L)$ is an isomorphism and there holds the following estimate*

$$\|(\lambda E + I)^{-1}\|_{B(L^p(0,L))} \leq (1+\sqrt{2})|\lambda|^{-1} \quad \textit{for} \quad \lambda \in \Sigma, \tag{2.12}$$

where

$$\Sigma = \{z \in \mathbb{C} : \ \operatorname{Re} z > |\operatorname{Im} z|\}.$$

Proof. Clearly, $\lambda E + I \in B(L^p(0,L))$, i.e. it is linear and bounded. We will show that the equivalence (2.11) defines a bounded inverse.

$$\begin{aligned}\|(\lambda E + I)^{-1}u\|_{L^p(0,L)} &= \|\lambda^{-1}u(t) - \lambda^{-2}\int_0^t u(s)e^{\frac{s-t}{\lambda}}ds\|_{L^p(0,L)}\\ &\leq |\lambda|^{-1}\|u\|_{L^p(0,L)} + |\lambda|^{-2}\|u\|_{L^p(0,L)}\|e^{-\frac{t}{\lambda}}\|_{L^1(0,L)},\end{aligned}$$

where we applied the Young inequality for convolution. Calculating the last norm we get

$$\|(\lambda E + I)^{-1}u\|_{L^p(0,L)} \leq |\lambda|^{-1}\left[1 + \frac{|\lambda|}{\operatorname{Re}\lambda}\left(1 - e^{-L\frac{\operatorname{Re}\lambda}{|\lambda|^2}}\right)\right]\|u\|_{L^p(0,L)}.$$

Applying the estimate

$$\frac{|\lambda|}{\operatorname{Re}\lambda}\left(1 - e^{L\frac{\operatorname{Re}\lambda}{|\lambda|^2}}\right) \leq \sqrt{2} \quad \text{for} \quad \lambda \in \Sigma,$$

we obtain (2.12). □

Propositions 2.19 and 2.20 show that I is non-negative operator in the sense of Definition 2.3. From Definition 2.6 we infer that positive powers of I are defined by Balakrishnan operator J^α given by Definition 2.5. Let us discuss here the case $0 < \operatorname{Re}\alpha < 1$. Then,

$$J^\alpha u = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-1}(\lambda E + I)^{-1}Iud\lambda.$$

By Proposition 2.19 we note that

$$(\lambda E + I)^{-1}Iu(t) = \lambda^{-1}\int_0^t u(\tau)d\tau - \lambda^{-2}\int_0^t\int_0^s u(\tau)d\tau e^{\frac{s-t}{\lambda}}ds = \lambda^{-1}\int_0^t u(\tau)e^{\frac{\tau-t}{\lambda}}d\tau.$$

Hence,

$$J^\alpha u = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-2}\int_0^t u(\tau)e^{\frac{\tau-t}{\lambda}}d\tau d\lambda.$$

Applying the Fubini theorem and then the substitution $\frac{t-\tau}{\lambda} = p$ we arrive at

$$J^\alpha u = \frac{\sin\alpha\pi}{\pi}\int_0^t u(\tau)(t-\tau)^{\alpha-1}\int_0^\infty p^{-\alpha}e^{-p}dpd\tau = \frac{\sin\alpha\pi}{\pi}\int_0^t u(\tau)(t-\tau)^{\alpha-1}d\tau\Gamma(1-\alpha).$$

Recalling that

$$\frac{\sin \pi\alpha}{\pi} = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}, \tag{2.13}$$

we obtain

$$J^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau)(t-\tau)^{\alpha-1} d\tau.$$

Proceeding similarly in the general case $\operatorname{Re}\alpha > 0$ we arrive at the following proposition.

Proposition 2.21. *Let* $\operatorname{Re}\alpha, L > 0$, $p \in [1,\infty]$. *Then the operator* I^α *defined in (2.9) as an operator acting on* $L^p(0,L)$ *coincides with the fractional power of integration operator defined by (2.10).*

As a matter of fact, this result is regarded as a powerful tool in the theory of fractional integrals. For instance, we may easily obtain the semigroup property $I^\alpha I^\beta = I^{\alpha+\beta}$. It may be shown that this formula is satisfied even if we consider only integrable functions.

Proposition 2.22. *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta, L > 0$, $f \in L^1(0,L)$. *Then, there holds*

$$I^\alpha I^\beta f = I^{\alpha+\beta} f.$$

The proof follows from the Fubini theorem and the following integral relation

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-a)^{\alpha+\beta-1} \text{ for } \operatorname{Re}\alpha, \operatorname{Re}\beta > 0, \ -\infty < a < b < \infty. \tag{2.14}$$

Now we will introduce the definitions of the Riemann-Liouville and Caputo fractional derivatives. Although, the fractional differential operators of arbitrary order $\alpha \in \mathbb{C}$, with $\operatorname{Re}\alpha > 0$ may be defined, in this work we focus our attention only on the case $0 < \operatorname{Re}\alpha < 1$.

Definition 2.18. *Let* $0 < \operatorname{Re}\alpha < 1$. *If* f *is regular enough (the discussion about appropriate regularity of* f *will be carried below) we may define the Riemann-*
-Liouville fractional derivative

$$\partial^\alpha f(x) = \frac{\partial}{\partial x} I^{1-\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x (x-p)^{-\alpha} f(p) dp$$

and the Caputo fractional derivative

$$D^\alpha f(x) = \frac{\partial}{\partial x}(I^{1-\alpha}[f(x) - f(0)]) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x (x-p)^{-\alpha}[f(p) - f(0)] dp.$$

It is clear that for functions belonging to $W^{1,1}(0,L)$ the foregoing fractional differential operators are well defined. Moreover, if f belongs to $W^{1,1}(0,L)$, then $D^\alpha f$ may be equivalently written in the form

$$D^\alpha f(x) = I^{1-\alpha} f'(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} f'(p) dp. \tag{2.15}$$

The proper definition of the domain of these operators seems to be challenging itself. Here, we will present a characterization of the domain of the Riemann-Liouville fractional derivative in L^p - spaces. Let us define the operator of differentiation

$$\frac{\partial}{\partial x} : D(\frac{\partial}{\partial x}) := {}_0W^{1,p}(0,L) \to L^p(0,L), \quad \frac{\partial}{\partial x}u := u' \quad \text{for} \quad p \in [1,\infty]. \tag{2.16}$$

Here, the space ${}_0W^{1,p}$ denotes the subspace of $W^{1,p}$ consisted of functions with vanishing trace at zero. We will establish the following proposition.

Proposition 2.23. *Let $L > 0$, $p \in [1,\infty]$, $0 < \operatorname{Re}\alpha < 1$. Then, the Riemann-Liouville fractional derivative ∂^α, defined by Definition 2.18, as an operator acting on $L^p(0,L)$ coincides with the fractional power of differentiation operator defined by (2.16).*

Proof. At first, we will show that $\frac{\partial}{\partial x}$ is a positive operator in the sense of Definition 2.4 on $L^p(0,L)$ for $p \in [1,\infty]$. To show nonnegativity of $\frac{\partial}{\partial x}$ we fix $v \in L^p(0,L)$ and we search for a solution to

$$\lambda u + \frac{\partial}{\partial x}u = v, \;\; \operatorname{Re}\lambda > 0,$$

belonging to $D(\frac{\partial}{\partial x})$. We multiply the equation by $e^{\lambda x}$.

$$\frac{\partial}{\partial x}(ue^{\lambda x}) = ve^{\lambda x}.$$

Since $u(0) = 0$, we get

$$u = \int_0^x e^{-\lambda(x-p)}v(p)dp \tag{2.17}$$

and by the Young inequality for convolution

$$\|u\|_{L^p(0,L)} \leq \|v\|_{L^p(0,L)} \left\|e^{-\lambda x}\right\|_{L^1(0,L)} \leq \frac{\|v\|_{L^p(0,L)}}{\operatorname{Re}\lambda} \quad \text{for} \quad \operatorname{Re}\lambda > 0.$$

Obviously, zero belongs to the resolvent set of $\frac{\partial}{\partial x}$ and $(\frac{\partial}{\partial x})^{-1} = I$, where I is an integration operator defined in (2.10). Hence, $\frac{\partial}{\partial x}$ is a positive operator in the sense of Definition 2.4. The fractional powers of $\frac{\partial}{\partial x}$ are defined due to Definition 2.7 in the following way

$$D((\frac{\partial}{\partial x})^\alpha) = \{u \in L^p(0,L) : u \in R(I^\alpha)\}$$

and

$$(\frac{\partial}{\partial x})^\alpha := (((\frac{\partial}{\partial x})^{-1})^\alpha)^{-1},$$

Making use of $(\frac{\partial}{\partial x})^{-1} = I$ and applying Definition 2.9, we arrive at

$$\left(\frac{\partial}{\partial x}\right)^\alpha = I^{-\alpha}.$$

On the other hand, if $u \in D(I^{-\alpha})$, then $u \in D(I^{1-\alpha}) = L^p(0,L)$. By Theorem 2.7 (applied with parameters $-1, 1-\alpha$) we obtain that $I^{1-\alpha}u \in D(I^{-1})$ and $I^{-\alpha}u = I^{-1}I^{1-\alpha}u$. Furthermore,

$$I^{-1}I^{1-\alpha}u = \frac{\partial}{\partial x}I^{1-\alpha}u = \partial^\alpha u.$$

Summing up the results, we obtain that

$$(\frac{\partial}{\partial x})^\alpha u = \partial^\alpha u \quad \text{for every} \quad u \in D((\frac{\partial}{\partial x})^\alpha) = D(I^{-\alpha}) = R(I^\alpha). \tag{2.18}$$

□

Our aim is to characterize the domain of ∂^α which coincides with the range of I^α in $L^p(0,L)$ for $\alpha \in (0,1)$ and $p \in (1,\infty)$. In order to do it, we recall the result concerning the boundedness of imaginary powers of $\frac{\partial}{\partial x}$.

Theorem 2.24. *[21, Theorem 12.1.9] Let $\frac{\partial}{\partial x}$ be defined by (2.16) and $p \in (1,\infty)$. Then,*

$$\left\| (\frac{\partial}{\partial x})^{i\tau} \right\|_{L^p(0,L)} \leq c(1+|\tau|)e^{\frac{\pi|\tau|}{2}} \quad for \quad \tau \neq 0,$$

where the imaginary powers are defined by Definition 2.9 and c is a positive constant which depends only on p.

We are ready to formulate and prove the results concerning the characterization of the domain of the Riemann-Liouville derivative in $L^p(0,L)$.

Proposition 2.25. *For $L > 0, \alpha \in (0,1), p \in (1,\infty)$ the operators $I^\alpha : L^p(0,L) \longrightarrow {}_0H^{\alpha,p}(0,L)$ and $\partial^\alpha : {}_0H^{\alpha,p}(0,L) \longrightarrow L^p(0,L)$ are isomorphism and the following inequalities hold*

$$c^{-1}\|u\|_{{}_0H^{\alpha,p}(0,L)} \leq \|\partial^\alpha u\|_{L^p(0,L)} \leq c\|u\|_{{}_0H^{\alpha,p}(0,L)} \quad for\ u \in {}_0H^{\alpha,p}(0,L),$$

$$c^{-1}\|I^\alpha f\|_{{}_0H^{\alpha,p}(0,L)} \leq \|f\|_{L^p(0,L)} \leq c\|I^\alpha f\|_{{}_0H^{\alpha,p}(0,L)} \quad for\ f \in L^p(0,L).$$

Here by ${}_0H^{\alpha,p}(0,L)$ we denote the fractional Lebesgue space defined by

$${}_0H^{\alpha,p}(0,L) := [L^p(0,L), {}_0W^{1,p}(0,L)]_\alpha$$

and c denotes a positive constant dependent on α, p, L.

Proof. Applying Theorem 2.24 together with Theorem 2.8 we obtain that the domain of $(\frac{\partial}{\partial x})^\alpha$ in $L^p(0,L)$ is given by ${}_0H^{\alpha,p}(0,L)$. Hence, by Proposition 2.23 we obtain that if we consider ∂^α as an operator acting on $L^p(0,L)$ we have $D(\partial^\alpha) = {}_0H^{\alpha,p}(0,L)$ with norm equivalence. Hence, $\|\partial^\alpha u\|_{L^p(0,L)} \leq c\,\|u\|_{{}_0H^{\alpha,p}(0,L)}$ for $u \in {}_0H^{\alpha,p}(0,L)$ and $\|I^\alpha f\|_{{}_0H^{\alpha,p}(0,L)} \leq c\,\|f\|_{L^p(0,L)}$ for $f \in L^p(0,L)$. The two remaining inequalities follows from Corollary 2.6. □

In this thesis we will work mainly in Hilbert spaces, hence the case of $p = 2$ in Proposition 2.25 is on particular interest. We will discuss this case in detail in Proposition 2.32. We finish this section with a remark concerning the Caputo derivative.

Remark 2.5. *Let $L > 0$ and $0 < \alpha < 1$. Let us discuss the operator $\frac{\partial}{\partial x}$ defined in (2.16). Then, the Balakrishnan operator J^α of $\frac{\partial}{\partial x}$ coincides with the Caputo derivative D^α defined in definition 2.18. Furthermore, the operator ∂^α defined on ${}_0H^{\alpha,p}(0,L)$ for $p \in (1,\infty)$ is the closure of D^α defined on ${}_0W^{1,p}(0,L)$.*

Proof. Let us calculate the Balakrishnan operator of $\frac{\partial}{\partial x}$. For $u \in D(\frac{\partial}{\partial x})$ we have

$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} \int_0^x e^{-\lambda(x-p)} u'(p) dp d\lambda = \frac{\sin \alpha\pi}{\pi} \int_0^x u'(p) \int_0^\infty \lambda^{\alpha-1} e^{-\lambda(x-p)} d\lambda dp.$$

Applying substitution $\lambda(x-p) = w$ and then the identities (2.13) and (2.15) we get

$$J^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} u'(p) dp = D^\alpha u,$$

hence, we obtained the first part of the statement. In view of Proposition 2.25, the second part of statement follows directly from Proposition 2.5. □

2.3. Properties of fractional operators

Now, we will present more elementary properties of fractional operators, that will be used further. For a comprehensive studies on this subject we refer to standard literature [11], [29].

Remark 2.6. *Directly from the definition we may note that the Riemann-Liouville and Caputo derivatives coincide for functions which vanish at zero. Moreover, for every absolutely continuous f there holds*

$$(D^\alpha f)(x) = (\partial^\alpha f)(x) - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} f(0).$$

Let us investigate how the fractional operators act on polynomial functions. This is a very simple but useful example.

Example 2.1. *Let $\alpha \in (0,1)$, $\beta > -1$. Then,*

$$I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}$$

and for $\beta > 0$

$$\partial^\alpha x^\beta = D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}.$$

In the forthcoming chapters we will notice that the behaviour of fractional operators acting on constant functions plays essential role in the theory of regularity of solutions to fractional-differential equations. We formulate this observation in the next example.

Example 2.2. *Let $\alpha > 0$. Then*

$$(I^\alpha 1)(x) = \frac{x^\alpha}{\Gamma(\alpha+1)} \quad \text{and for} \quad \alpha \in (0,1) \quad (\partial^\alpha 1)(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad (D^\alpha 1)(x) = 0.$$

We note that the Riemann-Liouville fractional derivative, unlike the Caputo derivative, does not vanish on constant functions. This rather unnatural behaviour of ∂^α is one of the reasons way the Caputo derivative is preferably used in many physical models.

As it was already mentioned, ∂^α is well defined for absolutely continuous functions. Proposition 2.25 gives us the characterization of the domain of ∂^α in L^p. However, there exist less regular functions such that their convolution with the kernel $x^{-\alpha}$ is absolutely continuous. For such functions ∂^α is well defined as well. Let us provide an example.

Example 2.3. *Let $\alpha \in (0,1)$. Applying (2.14) we obtain that*

$$\partial^\alpha x^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}p^{\alpha-1}dp = \frac{\partial}{\partial x}\Gamma(\alpha) = 0.$$

Obviously the function $x^{\alpha-1}$ is not absolutely continuous on $[0,1]$. Furthermore,

$$x^{\alpha-1} \notin [L^p(0,1), {}_0W^{1,p}(0,1)]_\alpha$$

for any $p \in (1,\infty)$. Indeed, we fix $p \in (1,\infty)$ and we consider the operator I^α defined on $L^p(0,1)$. In view of Proposition 2.25 it is enough to show that $x^{\alpha-1} \notin R(I^\alpha)$. Let us suppose that there exists $w \in L^p(0,1)$ such that $I^\alpha w = x^{\alpha-1}$. Then, $\partial^\alpha I^\alpha w = w$ and $\partial^\alpha x^{\alpha-1} = 0$. Hence $w = 0$, which leads to a contradiction with $I^\alpha w = x^{\alpha-1}$ because $I^\alpha 0 = 0$.

We note that, since $x^{\alpha-1}$ is not well defined at the origin, the Caputo fractional derivative for this function is not well defined.

Considering the fractional derivatives we can not expect the usual formula for differentiation of the product. However, we have the following result.

Proposition 2.26. *Let $L > 0$, $\alpha \in (0,1)$. If $f, g \in AC[0,L]$ and $g \in C^{0,\beta}([0,L])$ for $\beta \in (\alpha, 1)$, then*

$$\partial^\alpha(f\cdot g)(x) = g(x)(\partial^\alpha f)(x) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}(g(x)-g(p))f(p)dp.$$

Proof. Let us perform the calculations

$$\partial^\alpha(f\cdot g)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}f(p)g(p)dp$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\left[g(x)\int_0^x (x-p)^{-\alpha}f(p)dp\right] - \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}f(p)(g(x)-g(p))dp.$$

Since $|g(x)-g(p)| \le \|g\|_{C^{0,\beta}(0,L)}|x-p|^\beta$ we may differentiate the last integral and we obtain

$$\partial^\alpha(f\cdot g)(x) = g'(x)\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha}f(p)dp + g(x)\partial^\alpha f(x)$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}f(p)(g(x)-g(p))dp - g'(x)\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha}f(p)dp$$

$$= g(x)\partial^\alpha f(x) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}f(p)(g(x)-g(p))dp.$$

□

We introduce the definition of Mittag-Leffler functions. These functions play an important role in the theory of fractional calculus.

Definition 2.19. *Let $\mu, \nu \in \mathbb{R}$, $\nu > 0$, then we define*

$$E_{\nu,\mu}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu+n\nu)} \quad \text{and} \quad E_\nu(z) = E_{\nu,1}(z).$$

Mittag-Leffler functions for positive ν and μ are entire functions with respect to z. We may consider these functions as the generalizations of exponential functions, since $E_{1,1}(z) = e^z$. The function $E_\alpha(\lambda t^\alpha)$ plays a significant role while considering fractional differential equations because it satisfies

$$D^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha).$$

This identity may be easily proven applying term by term differentiation.
We present here, quite sophisticated result obtained in [24] which we will apply in the next chapter.

Proposition 2.27. *[24, Theorem 4.2.1] Let $E_{\nu,\mu}(\cdot)$ denotes the Mittag-Leffler function. If we suppose that either*

$$0 < \nu < 1, \ \ \mu \in [1, 1+\nu] \quad \textit{or} \quad \nu \in (1,2), \ \ \mu \in [\nu-1, 1] \cup [\nu, 2],$$

then all roots of the function $E_{\nu,\mu}$ lie outside the angle

$$|\arg z| \leq \frac{\pi\nu}{2}.$$

One of the fundamental issues, for solving the fractional differential equations with the Caputo derivative, is to investigate whether the operator I^α acts like an operator inverse to D^α. We cite here Lemma 2.21 from [11], however instead of the L^∞ assumption we assume L^p regularity.

Proposition 2.28. *[11, Lemma 2.21] Let $L > 0$, $\alpha \in (0,1)$. Then, we have*

$$(D^\alpha I^\alpha f)(x) = f(x) \quad \textit{for} \quad f \in L^p(0,L), \ \ p > \frac{1}{\alpha},$$
$$(I^\alpha D^\alpha f)(x) = f(x) - f(0) \quad \textit{for} \quad f \in AC[0,L].$$

Proof. We note that if $f \in L^p(0,L)$ for $p > \frac{1}{\alpha}$, then we may apply Hölder inequality with parameters p, $\frac{p}{p-1}$ to obtain

$$|I^\alpha f(x)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^x (x-p)^{\alpha-1} f(p) dp \right| \leq \|f\|_{L^p(0,L)}^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha)} \frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} x^{\frac{p\alpha-1}{p}} \xrightarrow{x\to 0} 0.$$

Hence, applying Remark 2.6, Definition 2.18 and Proposition 2.22 we arrive at

$$D^\alpha I^\alpha f = \partial^\alpha I^\alpha f = \frac{\partial}{\partial x} I^{1-\alpha} I^\alpha f = \frac{\partial}{\partial x} I f = f.$$

To show the second identity we take $f \in AC[0,L]$ and we apply formula (2.15) together with Proposition 2.22 to obtain

$$(I^\alpha D^\alpha f)(x) = I^\alpha I^{1-\alpha} f'(x) = I f'(x) = f(x) - f(0).$$

□

Now, we present an analogous result in the case of the fractional Riemann-Liouville derivative.

Proposition 2.29. *[29, Theorem 2.4] Let $\alpha \in (0,1)$ and $L > 0$. Then,*

$$\partial^\alpha I^\alpha f = f \quad \text{for} \quad f \in L^1(0,L).$$

If $f \in L^1(0,L)$ is such that $\partial^\alpha f \in L^1(0,L)$, then we have

$$I^\alpha \partial^\alpha f(x) = f(x) - \frac{x^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} f(0),$$

where

$$I^{1-\alpha} f(0) := \lim_{x \to 0} I^{1-\alpha} f(x).$$

We note that the limit is well defined because by the assumption $I^{1-\alpha} f$ is absolutely continuous. In particular, if f additionally belongs to $L^p(0,L)$ for $p > \frac{1}{1-\alpha}$ then,

$$I^\alpha \partial^\alpha f = f.$$

Proof. If f is integrable then by Proposition 2.22 we have

$$\partial^\alpha I^\alpha f = \frac{\partial}{\partial x} I^{1-\alpha} I^\alpha f = \frac{\partial}{\partial x} I f = f.$$

Under the assumption $\partial^\alpha f \in L^1(0,L)$ we may write

$$I^\alpha \partial^\alpha f = I^\alpha \frac{\partial}{\partial x} I^{1-\alpha} f = D^{1-\alpha} I^{1-\alpha} f = \partial^{1-\alpha}(I^{1-\alpha} f - I^{1-\alpha} f(0)),$$

where we used identity (2.15) and Definition 2.18. Applying the first part of the claim and Example 2.2 we arrive at

$$I^\alpha \partial^\alpha f(x) = f(x) - I^{1-\alpha} f(0) \frac{x^{\alpha-1}}{\Gamma(\alpha)}.$$

If additionally $f \in L^p(0,L)$ for $p > \frac{1}{1-\alpha}$ we obtain by Hölder inequality (as in the proof of Proposition 2.28) that $I^{1-\alpha} f(0) = 0$ and the proof is finished. □

We illustrate how does the Proposition 2.29 work on an example.

Example 2.4. *Let us discuss the function $x^{\alpha-1}$. Then, according to Proposition 2.29 we obtain that $\partial^\alpha I^\alpha x^{\alpha-1} = x^{\alpha-1}$. However, $I^\alpha \partial^\alpha x^{\alpha-1} = 0 = x^{\alpha-1} - \frac{x^{\alpha-1}}{\Gamma(\alpha)} \Gamma(\alpha)$, where $\Gamma(\alpha) = (I^{1-\alpha} x^{\alpha-1})(0)$.*

Below we present a simple proposition which gives us the formula for the superposition of ∂^α and D^α.

Proposition 2.30. *[15, Proposition 6.5] Let $L > 0$. For $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$ and $f \in AC[0,L]$ we have $\partial^\beta D^\alpha f = D^{\alpha+\beta} f$.*

Proof. Indeed, by Definition 2.18 and formula (2.15) we have

$$\partial^\beta D^\alpha f = \frac{\partial}{\partial x} I^{1-\beta} I^{1-\alpha} \frac{\partial}{\partial x} f = \frac{\partial}{\partial x} I^{2-(\beta+\alpha)} \frac{\partial}{\partial x} f,$$

where in the last identity we applied Proposition 2.22. Further, we get

$$\partial^\beta D^\alpha f = \frac{\partial}{\partial x} I I^{1-(\beta+\alpha)} \frac{\partial}{\partial x} f = I^{1-(\beta+\alpha)} \frac{\partial}{\partial x} f = D^{\alpha+\beta} f.$$

□

Remark 2.7. *Although the relation $\partial^\alpha \partial^\beta = \partial^{\alpha+\beta}$ is not true in general for absolutely continuous functions, we may show that for absolutely continuous f and $\alpha \in (0,1)$ there holds*

$$\partial^\alpha \partial^{1-\alpha} f = \frac{\partial}{\partial x} f.$$

Indeed, we apply firstly Remark 2.6 and then we use Example 2.3 to get

$$\partial^\alpha \partial^{1-\alpha} f = \partial^\alpha [D^{1-\alpha} f + \frac{x^{\alpha-1}}{\Gamma(1-\alpha)} f(0)] = \partial^\alpha D^{1-\alpha} f = \frac{\partial}{\partial x} f.$$

Analogously to the fractional operators defined in Definition 2.17 and Definition 2.18 we may consider right-side operators.

Definition 2.20. *Let $L > 0$. For $\operatorname{Re}\alpha > 0$ and $f \in L^1(0,L)$ we define*

$$I_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^L (p-x)^{\alpha-1} f(p) dp.$$

Analogously, for f regular enough and $0 < \operatorname{Re}\alpha < 1$ we may define

$$\partial_-^\alpha f(x) = -\frac{\partial}{\partial x}(I_-^{1-\alpha} f)(x) \text{ and } D_-^\alpha f(x) = -\frac{\partial}{\partial x} I_-^{1-\alpha}[f(x) - f(L)].$$

All the properties discussed above may be easily transferred for the case of I_-^α, ∂_-^α and D_-^α. We present here, the proposition from [15] which provides the energy estimate for the Riemann-Liouville fractional derivative. This estimate appears to be essential in further considerations.

Proposition 2.31. *[15, Proposition 6.10] If $w \in AC[0,L]$, then for any $\alpha \in (0,1)$ the following equality holds*

$$\int_0^L \partial^\alpha w(x) \cdot w(x) dx = \frac{\alpha}{4} \int_0^L \int_0^L \frac{|w(x) - w(p)|^2}{|x-p|^{1+\alpha}} dp dx$$
$$+ \frac{1}{2\Gamma(1-\alpha)} \int_0^L [(L-x)^{-\alpha} + x^{-\alpha}] |w(x)|^2 dx.$$

Hence, there exists a positive constant c which depends only on α, L, such that

$$\int_0^L \partial^\alpha w(x) \cdot w(x) dx \geq c \|w\|^2_{H^{\frac{\alpha}{2}}(0,L)} \tag{2.19}$$

and in particular

$$\int_0^L \partial^\alpha w(x) \cdot w(x) dx \geq \frac{L^{-\alpha}}{2\Gamma(1-\alpha)} \int_0^L |w(x)|^2 dx.$$

Proof. Let us perform the calculations. By Remark 2.6 we may write

$$\int_0^L \partial^\alpha w(x) \cdot w(x) dx = \int_0^L D^\alpha w(x) \cdot w(x) dx + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^L x^{-\alpha} w(x) dx.$$

Next, by definition of D^α we have

$$\int_0^L \partial^\alpha w(x) \cdot w(x) dx = \frac{1}{\Gamma(1-\alpha)} \int_0^L \int_0^x (x-p)^{-\alpha} w'(p) dp \cdot w(x) dx + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^L x^{-\alpha} w(x) dx$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_0^L \int_0^x (x-p)^{-\alpha} w'(p) \cdot [w(x) - w(p)] dp dx + \frac{w(0)}{\Gamma(1-\alpha)} \int_0^L x^{-\alpha} w(x) dx$$
$$+ \frac{1}{\Gamma(1-\alpha)} \int_0^L \int_0^x (x-p)^{-\alpha} w'(p) w(p) dp dx$$

$$= -\frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\int_0^L\int_0^x (x-p)^{-\alpha}\left(|w(x)-w(p)|^2\right)_p dpdx + \frac{w(0)}{\Gamma(1-\alpha)}\int_0^L x^{-\alpha}w(x)dx$$
$$+\frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\int_0^L\int_0^x (x-p)^{-\alpha}\left(|w(p)|^2\right)_p dpdx$$
$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^L\int_0^x \frac{|w(x)-w(p)|^2}{(x-p)^{\alpha+1}}dpdx - \frac{1}{2\Gamma(1-\alpha)}\int_0^L (x-p)^{-\alpha}|w(x)-w(p)|^2\Big|_{p=0}^{p=x}dt$$
$$+\frac{w(0)}{\Gamma(1-\alpha)}\int_0^L x^{-\alpha}w(x)dx + \frac{1}{2\Gamma(1-\alpha)}\int_0^L \left(|w(p)|^2\right)_p\int_p^L (x-p)^{-\alpha}dxdp.$$

Applying the Lebesgue integral differentiation theorem, we obtain that

$$(x-p)^{-1}\int_p^x w'(s)ds \overset{p\to x^-}{\longrightarrow} w'(x) \text{ for a.a. } x$$

and thus

$$\lim_{p\to x^-}(x-p)^{-\alpha}|w(x)-w(p)|^2 = \lim_{p\to x^-}(x-p)^{2-\alpha}\left|(x-p)^{-1}\int_p^x w'(s)ds\right|^2 = 0.$$

Hence, we obtain

$$\int_0^L \partial^\alpha w(x)\cdot w(x)dx =$$
$$\frac{\alpha}{2\Gamma(1-\alpha)}\int_0^L\int_0^x \frac{|w(x)-w(p)|^2}{(x-p)^{\alpha+1}}dpdx + \frac{1}{2\Gamma(1-\alpha)}\int_0^L x^{-\alpha}|w(x)-w(0)|^2dx$$
$$+\frac{w(0)}{\Gamma(1-\alpha)}\int_0^L x^{-\alpha}w(x)dx + \frac{1}{2\Gamma(2-\alpha)}\int_0^L (L-p)^{1-\alpha}\left(|w(p)|^2\right)_p dp$$
$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^L\int_0^x \frac{|w(x)-w(p)|^2}{(x-p)^{\alpha+1}}dpdx + \frac{1}{2\Gamma(1-\alpha)}\int_0^L x^{-\alpha}|w(x)-w(0)|^2dx$$
$$+\frac{w(0)}{\Gamma(1-\alpha)}\int_0^L x^{-\alpha}w(x)dx + \frac{1}{2}\frac{1}{\Gamma(1-\alpha)}\int_0^L (L-p)^{-\alpha}|w(p)|^2dp - \frac{1}{2}\frac{1}{\Gamma(2-\alpha)}L^{1-\alpha}|w(0)|^2$$
$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^L\int_0^x \frac{|w(x)-w(p)|^2}{(x-p)^{\alpha+1}}dpdx + \frac{1}{2\Gamma(1-\alpha)}\int_0^L (L-p)^{-\alpha}|w(p)|^2dp$$
$$+\frac{1}{2\Gamma(1-\alpha)}\int_0^L p^{-\alpha}|w(p)|^2dp$$

and the proof is finished. □

Now, we will discuss in detail the results of Proposition 2.25 in the case $p=2$. We note that for $p=2$ an alternative proof of Proposition 2.25 was given in [9]. At first let us introduce the following functional spaces

$$_0H^\alpha(0,1) = \begin{cases} H^\alpha(0,1) & \text{for} \quad \alpha\in(0,\frac{1}{2}), \\ \{u\in H^{\frac{1}{2}}(0,1):\ \int_0^1\frac{|u(x)|^2}{x}dx<\infty\} & \text{for} \quad \alpha=\frac{1}{2}, \\ \{u\in H^\alpha(0,1):\ u(0)=0\} & \text{for} \quad \alpha\in(\frac{1}{2},1) \end{cases}$$

and

$$^0H^\alpha(0,1) = \begin{cases} H^\alpha(0,1) & \text{for} \quad \alpha\in(0,\frac{1}{2}), \\ \{u\in H^{\frac{1}{2}}(0,1):\ \int_0^1\frac{|u(x)|^2}{1-x}dx<\infty\} & \text{for} \quad \alpha=\frac{1}{2}, \\ \{u\in H^\alpha(0,1):\ u(1)=0\} & \text{for} \quad \alpha\in(\frac{1}{2},1). \end{cases}$$

We set $\|u\|_{{}_0H^\alpha(0,1)} = \|u\|_{{}^0H^\alpha(0,1)} = \|u\|_{H^\alpha(0,1)}$ for $\alpha \neq \frac{1}{2}$ and

$$\|u\|_{{}_0H^{\frac{1}{2}}(0,1)} = \left(\|u\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{|u(x)|^2}{x} dx\right)^{\frac{1}{2}},$$

$$\|u\|_{{}_0H^{\frac{1}{2}}(0,1)} = \left(\|u\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{|u(x)|^2}{1-x} dx\right)^{\frac{1}{2}}.$$

The spaces defined above may be equivalently defined in terms of complex interpolation spaces, i.e.

$${}_0H^\alpha(0,1) = [L^2(0,1), {}_0H^1(0,1)]_\alpha \quad \text{and} \quad {}^0H^\alpha(0,1) = [L^2(0,1), {}^0H^1(0,1)]_\alpha.$$

Here, by ${}_0H^1(0,1)$ we understand the subspace of $H^1(0,1)$ consisting of functions which trace vanishes at the left endpoint of the interval. Analogously, ${}^0H^1(0,1)$ is a subspace of $H^1(0,1)$ consisting of functions which trace vanishes at the right endpoint of the interval. The following proposition is the special case of Proposition 2.25. It can be found also in the Appendix of [15] as an extended version of [9, Theorem 2.1].

Proposition 2.32. *For $\alpha \in [0,1]$ the operators $I^\alpha : L^2(0,1) \longrightarrow {}_0H^\alpha(0,1)$ and $\partial^\alpha : {}_0H^\alpha(0,1) \longrightarrow L^2(0,1)$ are isomorphism and the following inequalities hold*

$$c_\alpha^{-1}\|u\|_{{}_0H^\alpha(0,1)} \leq \|\partial^\alpha u\|_{L^2(0,1)} \leq c_\alpha \|u\|_{{}_0H^\alpha(0,1)} \quad \text{for } u \in {}_0H^\alpha(0,1),$$

$$c_\alpha^{-1}\|I^\alpha f\|_{{}_0H^\alpha(0,1)} \leq \|f\|_{L^2(0,1)} \leq c_\alpha \|I^\alpha f\|_{{}_0H^\alpha(0,1)} \quad \text{for } f \in L^2(0,1).$$

Analogously, by the change of variables $x \mapsto 1-x$, we obtain that the operators $I^\alpha_- : L^2(0,1) \longrightarrow {}^0H^\alpha(0,1)$ and $\partial^\alpha_- : {}^0H^\alpha(0,1) \longrightarrow L^2(0,1)$ are isomorphism and there hold the inequalities

$$c_\alpha^{-1}\|u\|_{{}^0H^\alpha(0,1)} \leq \|\partial^\alpha_- u\|_{L^2(0,1)} \leq c_\alpha \|u\|_{{}^0H^\alpha(0,1)} \quad \text{for } u \in {}^0H^\alpha(0,1),$$

$$c_\alpha^{-1}\|I^\alpha_- f\|_{{}^0H^\alpha(0,1)} \leq \|f\|_{L^2(0,1)} \leq c_\alpha \|I^\alpha_- f\|_{{}^0H^\alpha(0,1)} \quad \text{for } f \in L^2(0,1).$$

Here c_α denotes a positive constant dependent on α.

Corollary 2.33. *For $\alpha, \beta > 0$ there holds $I^\beta : {}_0H^\alpha(0,1) \to {}_0H^{\alpha+\beta}(0,1)$, where in the case $\gamma > 1$*

$${}_0H^\gamma(0,1) = \{f \in H^\gamma(0,1) : f^{(k)}(0) = 0, \ k = 0, \ldots, \lfloor\gamma\rfloor - 1, \ f^{(\lfloor\gamma\rfloor)} \in {}_0H^{\gamma-\lfloor\gamma\rfloor}(0,1)\}.$$

Furthermore, there exists a positive constant c dependent only on α, β such that for every $f \in {}_0H^\alpha(0,1)$

$$\left\|I^\beta f\right\|_{{}_0H^{\alpha+\beta}(0,1)} \leq c\,\|f\|_{{}_0H^\alpha(0,1)}.$$

Proof. It is an easy consequence of Proposition 2.32. If $f \in {}_0H^\alpha(0,1)$ then, $f^{(\lfloor\alpha\rfloor)} \in {}_0H^{\alpha-\lfloor\alpha\rfloor}(0,1)$. By Proposition 2.32, there exists $w \in L^2(0,1)$ such that $f^{(\lfloor\alpha\rfloor)} = I^{\alpha-\lfloor\alpha\rfloor}w$. Hence, applying Proposition 2.22 we get

$$f = I^{\lfloor\alpha\rfloor} f^{(\lfloor\alpha\rfloor)} = I^{\lfloor\alpha\rfloor} I^{\alpha-\lfloor\alpha\rfloor} w = I^\alpha w.$$

$$I^\beta f = I^\beta I^\alpha w = I^{\alpha+\beta} w = I^{\lfloor \beta+\alpha \rfloor} I^{\beta+\alpha-\lfloor \beta+\alpha \rfloor} w. \tag{2.20}$$

If $\alpha + \beta \leq 1$, then applying again Proposition 2.32 we obtain that $I^\beta f \in {}_0H^{\alpha+\beta}(0,1)$. Moreover, we note that $w = \partial^\alpha f$ and by Proposition 2.32 we have

$$\left\| I^\beta f \right\|_{{}_0H^{\alpha+\beta}(0,1)} = \left\| I^{\alpha+\beta} w \right\|_{{}_0H^{\alpha+\beta}(0,1)} \leq c(\alpha,\beta) \left\| w \right\|_{L^2(0,1)} \leq c(\alpha,\beta) \left\| f \right\|_{{}_0H^\alpha(0,1)}.$$

In the case $1 < \alpha + \beta$ from (2.20) we infer that

$$(I^\beta f)^{(k)}(0) = 0 \text{ for } k = 0, \dots, \lfloor \beta + \alpha \rfloor - 1 \tag{2.21}$$

and by Proposition 2.32

$$(I^\beta f)^{(\lfloor \beta+\alpha \rfloor)} = I^{\beta+\alpha-\lfloor \beta+\alpha \rfloor} w \in {}_0H^{\beta+\alpha-\lfloor \beta+\alpha \rfloor}(0,1).$$

Due to (2.21) we may apply Poincaré inequality to obtain

$$\begin{aligned} \left\| I^\beta f \right\|_{{}_0H^{\alpha+\beta}(0,1)} &\leq c \left\| (I^\beta f)^{(\lfloor \beta+\alpha \rfloor)} \right\|_{{}_0H^{\alpha+\beta-\lfloor \alpha+\beta \rfloor}(0,1)} = \left\| I^{\beta+\alpha-\lfloor \beta+\alpha \rfloor} w \right\|_{{}_0H^{\alpha+\beta-\lfloor \alpha+\beta \rfloor}(0,1)} \\ &\leq c \left\| w \right\|_{L^2(0,1)} \leq c \left\| \partial^{\alpha-\lfloor \alpha \rfloor} f^{(\lfloor \alpha \rfloor)} \right\|_{L^2(0,1)} \leq c \left\| f^{(\lfloor \alpha \rfloor)} \right\|_{{}_0H^{\alpha-\lfloor \alpha \rfloor}(0,1)} \leq c \left\| f \right\|_{{}_0H^\alpha(0,1)}, \end{aligned}$$

where we applied Proposition 2.32 and $c = c(\alpha, \beta)$.

□

In subsequent parts of this thesis we will make use of the following local property established in [28].

Lemma 2.34. *Let $f \in {}_0H^\alpha(0,1)$ for $\alpha \in (0,1)$ and $\partial^\alpha f \in H^\beta_{loc}(0,1)$ for $\beta \in (\frac{1}{2}, 1]$. Then $f \in H^{\beta+\alpha}_{loc}(0,1)$ and for every $0 < \delta < \omega < 1$ there exists a positive constant $c = c(\delta, \omega, \alpha, \beta)$ such that*

$$\|f\|_{H^{\beta+\alpha}(\delta,\omega)} \leq c(\|f\|_{{}_0H^\alpha(0,\omega)} + \|\partial^\alpha f\|_{H^\beta(\frac{\delta}{2},\omega)}). \tag{2.22}$$

Proof. Let us fix $0 < \delta < \omega < 1$. Then, by the assumption we have $\partial^\alpha f \in H^\beta(\frac{\delta}{2}, \omega)$. Applying Proposition 2.32, for $x > \delta/2$ we may write

$$\begin{aligned} f(x) &= I^\alpha \partial^\alpha f(x) = I^\alpha(\partial^\alpha f - \partial^\alpha f(\delta/2))(x) + I^\alpha(\partial^\alpha f(\delta/2))(x) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\frac{\delta}{2}} (x-p)^{\alpha-1} (\partial^\alpha f(p) - \partial^\alpha f(\delta/2)) dp + \frac{1}{\Gamma(\alpha)} \int_{\frac{\delta}{2}}^x (x-p)^{\alpha-1} (\partial^\alpha f(p) - \partial^\alpha f(\delta/2)) dp \\ &\quad + \frac{1}{\Gamma(\alpha)} \partial^\alpha f(\delta/2) \int_0^{\frac{\delta}{2}} (x-p)^{\alpha-1} dp + \frac{1}{\Gamma(\alpha)} \partial^\alpha f(\delta/2) \int_{\frac{\delta}{2}}^x (x-p)^{\alpha-1} dp \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\frac{\delta}{2}} (x-p)^{\alpha-1} \partial^\alpha f(p) dp + I^\alpha_{\frac{\delta}{2}} (\partial^\alpha f - \partial^\alpha f(\delta/2))(x) + \frac{1}{\Gamma(1+\alpha)} (x - \delta/2)^\alpha \partial^\alpha f(\delta/2), \end{aligned}$$

where we applied Definition 2.17 and we denoted $I^\alpha_a f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-p)^{\alpha-1} f(p) dp$. We note that by Corollary 2.32, the second component belongs to ${}_0H^{\alpha+\beta}(\frac{\delta}{2}, \omega)$. The first and third component belong to $H^{\alpha+\beta}(\delta, \omega)$ because they are smooth on the interval (δ, ω). Thus $f \in H^{\alpha+\beta}(\delta, \omega)$. Moreover,

$$\|f\|_{H^{\alpha+\beta}(\delta,\omega)} \leq c(\alpha,\beta,\delta,\omega)[\|\partial^\alpha f\|_{L^2(0,\omega)} + \|\partial^\alpha f\|_{{}_0H^\beta(\frac{\delta}{2},\omega)} + |\partial^\alpha f(\delta/2)|].$$

To show the estimate (2.22) it is enough to apply Proposition 2.32 together with the Sobolev estimate $|\partial^\alpha f(\delta/2)| \leq c \, \|\partial^\alpha f\|_{H^\beta(\delta/2,w)}$ which holds because $\beta > \frac{1}{2}$.

$\square$

We finish this section with two propositions from [27] which provide us an extension of I^α and ∂^α into wider functional spaces. The similar reasoning to the one carried in Proposition 2.35 may be found in [7, Lemma 5].

Proposition 2.35. *For $\alpha \in (0, \frac{1}{2})$ the operators I^α and I^α_- can be extended to bounded and linear operators from $H^{-\alpha}(0,1) := (H^\alpha_0(0,1))'$ to $L^2(0,1)$.*

Here zero in the lower right index denotes vanishing trace at the boundary.

Proof. We will prove the claim only for I^α while the proof for I^α_- is analogous. By the Fubini theorem for $u, v \in L^2(0,1)$ we obtain

$$(I^\alpha u, v) = \left(u, I^\alpha_- v\right). \tag{2.23}$$

Applying Proposition 2.32 we obtain that $I^\alpha_- v \in {}^0H^\alpha(0,1)$ and we may estimate

$$|(I^\alpha u, v)| \leq \left\|I^\alpha_- v\right\|_{H^\alpha(0,1)} \|u\|_{(H^\alpha(0,1))'} \leq c_\alpha \|v\|_{L^2(0,1)} \|u\|_{H^{-\alpha}(0,1)},$$

where we used the fact that for $\alpha < \frac{1}{2}$ we have $H^\alpha_0(0,1) = H^\alpha(0,1)$ and thus $(H^\alpha_0(0,1))' = (H^\alpha(0,1))'$. The last inequality finishes the proof. $\square$

Proposition 2.36. *For $\alpha \in (0, \frac{1}{2})$ the operators ∂^α and ∂^α_- can be extended to bounded and linear operators from $L^2(0,1)$ to $H^{-\alpha}(0,1)$.*

Proof. As in the previous proposition, we will prove the statement only for ∂^α, because in the case ∂^α_- the proof is analogous. Let us assume that $f, v \in H^\alpha(0,1)$. (We recall that for $\alpha \in (0, \frac{1}{2})$ the space $H^\alpha(0,1)$ coincides with ${}^0H^\alpha(0,1)$ and ${}_0H^\alpha(0,1)$). Then, from Proposition 2.32, there exist $g \in L^2(0,1)$ such that $\partial^\alpha f = g$ and $w \in L^2(0,1)$ such that $v = I^\alpha_- w$. Thus, we have

$$(\partial^\alpha f, v) = \left(g, I^\alpha_- w\right) = (I^\alpha g, w) = \left(f, \partial^\alpha_- v\right).$$

Making use of Proposition 2.32 one more time, we may estimate

$$|(\partial^\alpha f, v)| \leq \|f\|_{L^2(0,1)} \left\|\partial^\alpha_- v\right\|_{L^2(0,1)} \leq c_\alpha \|f\|_{L^2(0,1)} \|v\|_{H^\alpha(0,1)},$$

Hence, the identity $(\partial^\alpha f, v) = \left(f, \partial^\alpha_- v\right)$ extends ∂^α to a bounded and linear operator from $L^2(0,1)$ to $(H^\alpha(0,1))'$. Since the space $(H^\alpha(0,1))'$ coincides with $H^{-\alpha}(0,1)$ for $\alpha \in (0, \frac{1}{2})$ the proof is finished. $\square$

2.4. Derivation of space-fractional Stefan model

In the next chapter we will investigate the properties of the operator $\frac{\partial}{\partial x} D^\alpha$. To motivate our study we discuss here the free boundary problem of space-fractional diffusion which was proposed in [32]. Here, we present a derivation of the model. The solution to this problem will be obtained in Chapter 4.

We will use a terminology of a heat transfer and the phenomenon of changing the phase of medium from solid to liquid. However, the following model may describe other anomalous diffusion processes as well, for example the mass transport and solidification of substances in complex media.

We consider the domain $(0, \infty)$. We assume that at the initial time $t = 0$ the domain is divided into two parts $(0, b)$, which can be regarded as liquid, and (b, ∞), which can be regarded as solid. We define the enthalpy function by $E = u + \varphi$. Here $u(x,t)$ denotes temperature of medium at point x in time t and φ denotes the latent heat. We consider a one-phase problem, hence we assume that $u \equiv 0$ at solid. Furthermore, we consider sharp-interphase problem. This means that the function φ has the form

$$\varphi = \begin{cases} 1 & \text{in liquid,} \\ 0 & \text{in solid.} \end{cases}$$

We denote by $q(x,t)$ the flux at point x at time t and we assume the following non-local form of the flux

$$q(x,t) = \begin{cases} -D^\alpha u(x,t) & \text{in liquid,} \\ 0 & \text{in solid.} \end{cases} \tag{2.24}$$

We note that since we discuss one-phase problem we have to put zero flux in the solid domain. In this setting the principle of energy conservation takes the following form. For every $(a, d) \subseteq (0, \infty)$ there hold

$$\frac{d}{dt} \int_a^d E(x,t)dx = q(a,t) - q(d,t). \tag{2.25}$$

We will derive the space-fractional Stefan problem from the formulas (2.24) and (2.25). Let us denote by $s(t)$ the interface. Then, at time t the liquid occupies $(0, s(t))$ and $(s(t), \infty)$ belongs to the solid domain. Let us fix $T > 0$ an arbitrary time. We denote the space-time domain occupied by the liquid by

$$Q_{s,T} := \{(x,t) : 0 < x < s(t),\ \ 0 < t < T\}.$$

In order to derive the model, we impose the following regularity properties

$$s \in AC[0,T],\ \ u_t(\cdot, t) \in L^1(0, s(t)) \quad \text{for every } t \in (0,T), \tag{2.26}$$

$$D^\alpha u(\cdot, t) \in C[0, s(t)] \cap AC_{loc}(0, s(t)) \quad \text{for every } t \in (0,T). \tag{2.27}$$

We may verify that the solution (u, s) obtained in Theorem 4.1 in Chapter 4 satisfies the assumptions above. In fact, it has higher regularity. We take $(a, t), (d, t) \in Q_{s,T}$, such that $a < d$ and we apply conservation law formula (2.25) to get

$$\frac{d}{dt} \int_a^d E(x,t)dx = q(a,t) - q(d,t).$$

Hence,

$$\frac{d}{dt} \int_a^d u(x,t) + 1dx = D^\alpha u(d,t) - D^\alpha u(a,t).$$

Under assumption (2.27) we may write

$$\int_a^d u_t(x,t)dx = \int_a^d \frac{\partial}{\partial x} D^\alpha u(x,t)dx.$$

Since the interval (a, d) was arbitrary we obtain

$$u_t = \frac{\partial}{\partial x} D^\alpha u \quad \text{in } Q_{s,T}.$$

In order to obtain an equation for the interface $s(\cdot)$ we take arbitrary $(a,t) \in Q_{s,T}$ and arbitrary $d > s(t)$. Then we may write the conservation law for an interval (a, d)

$$\frac{d}{dt} \int_a^d E(x,t)dx = q(a,t) - q(d,t).$$

Since $q(d, t) = 0$ we have

$$\frac{d}{dt} \int_a^{s(t)} u(x,t) + 1dx + \frac{d}{dt} \int_{s(t)}^d u(x,t)dx = -D^\alpha u(a,t).$$

We assume that at the free boundary the medium is in the phase-change temperature, i. e. $u(s(t), t) = 0$ and the function u vanishes in solid, thus we arrive at

$$\int_a^{s(t)} u_t(x,t)dx + \dot{s}(t) = -D^\alpha u(a,t).$$

Passing with a to $s(t)$ and applying the assumptions (2.26) and (2.27) we get

$$\dot{s}(t) = -D^\alpha u(s(t), t) \quad \text{in } (0, T).$$

We complement our system with initial and boundary conditions. On the left endpoint of the interval we assume zero Neumann boundary condition, however other options are also possible. Finally, we obtain the system of equations

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } Q_{s,T}, \\ u_x(0,t) = 0, \quad u(t, s(t)) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } 0 < x < s(0) = b, \\ \dot{s}(t) = -(D^\alpha u)(s(t), t) & \text{for } t \in (0,T). \end{cases} \tag{2.28}$$

We will solve this problem in Chapter 4.

2.5. Derivation of time-fractional Stefan model

In this section we will derive the one-phase time-fractional Stefan model. We consider the same setting as in the previous section, however now, following [33], we assume that the flux is given by the Riemann-Liouville fractional derivative with respect to the time variable, i.e.

$$q(x,t) = -\partial^{1-\alpha} u_x(x,t) = -\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{\alpha-1} u_x(x,\tau)d\tau. \tag{2.29}$$

The derivation of model presented in this section comes from [14]. We will proceed as in the previous section, i.e. we will derive the model from energy conservation formula (2.25), however now the diffusive flux is given by (2.29). Again we will use the terminology of the heat transfer, however we note that the model may be applied also to other anomalous diffusion processes. In order to derive the model rigorously we have to impose certain regularity conditions on the interface s and the temperature function u. We fix $t^* > 0$ and we denote

$$Q_{s,t^*} = \{(x,t) : \ \ 0 < x < s(t), \ \ t \in (0,t^*)\}.$$

The standard setting of the initial-boundary conditions for the Stefan problem is the following

$$u(x,0) = u_0(x) \geq 0 \text{ and } \ \ u(0,t) = u_D(t) \geq 0 \ \ \text{ or } \ \ u_x(0,t) = u_N(t) \leq 0.$$

We expect that if $u_0 \equiv 0$, $u_D \equiv 0$ or $u_0 \equiv 0$, $u_N \equiv 0$, then $u \equiv 0$. Otherwise, we expect

$$\dot{s}(t) > 0, \tag{A1}$$

i.e. melting of solid. Secondly, we assume that

$$s \in AC[0,t^*], \ \ u_x(x,\cdot) \in AC[s^{-1}(x),t^*] \text{ for every } x \in (0,s(t^*)),$$

$$u_x(\cdot,t) \in AC[0,s(t)-\varepsilon] \text{ for every } \varepsilon > 0 \text{ and every } t \in (0,t^*), \tag{A2}$$

$$u_t(\cdot,t) \in L^1(0,s(t)) \quad \text{for each } t \in (0,t^*).$$

We note that since we consider one-phase Stefan problem the temperature in the solid vanishes. Therefore, the flux is nonzero only in the liquid part of the domain, i.e. in Q_{s,t^*} and it is given by the formula

$$q(x,t) = \begin{cases} -\partial^{1-\alpha}_{s^{-1}(x)} u_x(x,t) & \text{for} \quad (x,t) \in Q_{s,t^*}, \\ 0 & \text{for} \quad (x,t) \notin Q_{s,t^*}, \end{cases} \tag{2.30}$$

where

$$\partial^{1-\alpha}_{s^{-1}(x)} u_x(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{\alpha-1} u_x(x,\tau)d\tau & \text{for} \quad x \leq s(0), \\ \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_{s^{-1}(x)}^t (t-\tau)^{\alpha-1} u_x(x,\tau)d\tau & \text{for} \quad x > s(0). \end{cases} \tag{2.31}$$

The last of the regularity assumptions, that we will take advantage of, are

$$\dot{s}(t) \in L^\infty_{loc}((0,t^*]) \text{ and } \ \ D^\alpha_{s^{-1}(x)} u(\cdot,t) \in L^1(0,s(t)) \text{ for } t \in (0,t^*). \tag{A3}$$

Now we are ready to formulate the result.

Theorem 2.37. *Let us discuss the sharp one-phase one-dimensional Stefan problem with the boundary condition $u(s(t),t)=0$. Then, under the assumptions (A1)-(A2), the conservation law (2.25) with the flux given by (2.29) leads to the following equation*

$$D^{\alpha}_{s^{-1}(x)}u(x,t)-u_{xx}(x,t)=\begin{cases} 0 & \text{for} \quad x<s(0), \\ -\frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} & \text{for} \quad x\in(s(0),s(t)) \end{cases} \tag{2.32}$$

for a.a. $(x,t)\in Q_{s,t^*}$, where

$$D^{\alpha}_{s^{-1}(x)}u(x,t)=\begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}\frac{d}{d\tau}u(x,\tau)d\tau & \text{for} \quad x\le s(0), \\ \frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(x)}^t(t-\tau)^{-\alpha}\frac{d}{d\tau}u(x,\tau)d\tau & \text{for} \quad x>s(0). \end{cases} \tag{2.33}$$

Moreover, functions u and s are related by the formula

$$\dot{s}(t)=-\lim_{a\nearrow s(t)}\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t)=-\frac{1}{\Gamma(\alpha)}\lim_{a\nearrow s(t)}\left[\frac{d}{dt}\int_{s^{-1}(a)}^t(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]. \tag{2.34}$$

Furthermore, if $(A3)$ holds, then the additional boundary condition

$$u_x^-(s(t),t):=\lim_{\varepsilon\to0^+}u_x^-(s(t)-\varepsilon,t)=0 \tag{2.35}$$

is satisfied.

We note that the equation (2.32) with the condition (2.34) have been already obtained in [25]. However, here we obtained the additional relation (2.35).

Proof. In order to derive the system of equations from (2.25), we apply the principle of energy conservation to an arbitrary subset V of the domain at time $t\in(0,t^*)$. We will consider two cases.

— If $V=(a,b)\subseteq(0,s(0))$, then from (A1) we have $V\subseteq(0,s(t))$ for each $t\in(0,t^*)$ and (2.25) gives

$$\frac{d}{dt}\left[\int_V u(x,t)+1dx\right]=\partial^{1-\alpha}u_x(b,t)-\partial^{1-\alpha}u_x(a,t).$$

Hence,

$$\int_V\frac{d}{dt}u(x,t)dx=\partial^{1-\alpha}u_x(b,t)-\partial^{1-\alpha}u_x(a,t).$$

We apply the fractional integral $I^{1-\alpha}$ with respect to the time variable to both sides of the identity and with a use of assumption (A2) we arrive at

$$\int_V D^{\alpha}u(x,t)dx=u_x(b,t)-u_x(a,t).$$

Indeed, we note that since u_x is absolutely continuous with respect to time we may apply Proposition 2.29 to get

$$I^{1-\alpha}\partial^{1-\alpha}u_x(x,t)=u_x(x,t).$$

By the fundamental theorem of calculus we obtain

$$\int_V[D^{\alpha}u(x,t)-u_{xx}(x,t)]dx=0.$$

Since $V\subseteq(0,s(0))$ is arbitrary, we get

$$D^{\alpha}u(x,t)-u_{xx}(x,t)=0 \quad \text{for} \quad (x,t)\in(0,s(0))\times(0,t^*). \tag{2.36}$$

— If $V=(a,b)$, where $s(0)<a<s(t)<b$, then (2.25) has the form

$$\frac{d}{dt}\left[\int_a^{s(t)} u(x,t)+1dx\right] = q(a,t) = -\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t).$$

Differentiating the integral on the left hand side leads to

$$\int_a^{s(t)} \frac{d}{dt}u(x,t)dx + \dot{s}(t)[u(s(t),t)+1] = -\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t).$$

Applying $u(s(t),t)=0$, we get

$$\int_a^{s(t)} \frac{d}{dt}u(x,t)dx + \dot{s}(t) = -\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t). \tag{2.37}$$

If $a \nearrow s(t)$, then by the assumption $(A2)$ the first term vanishes and as a consequence we arrive at (2.34). Next, if we apply the operator $I^{1-\alpha}_{s^{-1}(a)}$ to both sides of (2.37), then we obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\int_a^{s(\tau)} \frac{d}{d\tau}u(x,\tau)dxd\tau + \frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau$$

$$= -I^{1-\alpha}_{s^{-1}(a)}\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t). \tag{2.38}$$

We note that by the assumption (A2) we have that $u_x(a,\cdot)\in AC[s^{-1}(a),t^*]$ hence, by Proposition 2.29, we get

$$I^{1-\alpha}_{s^{-1}(a)}\partial^{1-\alpha}_{s^{-1}(a)}u_x(a,t) = u_x(a,t).$$

If we apply the Fubini theorem to the first term in (2.38), then we arrive at the identity

$$\int_a^{s(t)} D^{\alpha}_{s^{-1}(x)}u(x,t)dx + \frac{1}{\Gamma(1-\alpha)}\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau = -u_x(a,t). \tag{2.39}$$

Applying the substitution $\tau = s^{-1}(x)$ we get

$$\int_{s^{-1}(a)}^t (t-\tau)^{-\alpha}\dot{s}(\tau)d\tau = \int_a^{s(t)} (t-s^{-1}(x))^{-\alpha}dx.$$

We allow that $u_x(\cdot,t)$ may admit singular behaviour near the phase change point. Thus, we proceed very carefully. We fix $\varepsilon>0$ such that $a<s(t)-\varepsilon$, then, by (A2) we have

$$-u_x(a,t) = \int_a^{s(t)-\varepsilon} u_{xx}(x,t)dx - u_x(s(t)-\varepsilon,t).$$

Making use of this identity in (2.39) we obtain

$$\int_a^{s(t)-\varepsilon}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) - u_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx$$

$$= -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - u_x(s(t)-\varepsilon,t). \tag{2.40}$$

Let us choose arbitrary $\tilde{a}$ such that $s(0)<\tilde{a}<a$. Repeating the above calculations for $\tilde{a}$ instead of a, we obtain that

$$\int_{\tilde{a}}^{s(t)-\varepsilon}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) - u_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx$$

$$= -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - u_x(s(t)-\varepsilon,t). \tag{2.41}$$

Subtracting the sides of (2.40) and (2.41) we arrive at

$$\int_{\tilde{a}}^{a}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) - u_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx = 0 \tag{2.42}$$

for arbitrary $a, \tilde{a} \in (s(0), s(t)-\varepsilon)$ hence, we may deduce that

$$D^{\alpha}_{s^{-1}(x)}u(x,t) - u_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} = 0 \quad \text{for} \quad x \in (s(0), s(t)), \tag{2.43}$$

i.e. (2.32) is proven.

It remains to show (2.35). From (2.41) and (2.43) we infer that

$$0 = -\int_{s(t)-\varepsilon}^{s(t)}\left[D^{\alpha}_{s^{-1}(x)}u(x,t) + \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha}\right]dx - u_x(s(t)-\varepsilon, t).$$

In order to obtain additional information about $u_x(s(t),t)$, we employ further regularity assumptions. Applying (A3) we immediately get

$$\lim_{\varepsilon\to 0^+}\int_{s(t)-\varepsilon}^{s(t)}(t-s^{-1}(x))^{-\alpha}dx = 0 \text{ and } \lim_{\varepsilon\to 0^+}\int_{s(t)-\varepsilon}^{s(t)}D^{\alpha}_{s^{-1}(x)}u(x,t)dx = 0. \tag{2.44}$$

Making use of (2.44) we obtain

$$\lim_{\varepsilon\to 0^+} u_x(s(t)-\varepsilon, t) = 0, \tag{2.45}$$

hence, we arrive at (2.35), which finishes the proof of Theorem 2.37.

□

We will find a special solution to the system obtained in Theorem 2.37 in Chapter 5.

Chapter 3

Operator $\frac{\partial}{\partial x}D^\alpha$ as a generator of an analytic semigroup

In this chapter we investigate the operator $\frac{\partial}{\partial x}D^\alpha$ from the perspective of operator theory. We will proceed as follows. At first, we will characterize the domain of $\frac{\partial}{\partial x}D^\alpha$ in $L^2(0,1)$. Then, we will show that $\frac{\partial}{\partial x}D^\alpha$ generates a C_0 -semigroup of contractions. Finally, we will prove, by an appropriate estimate of the resolvent operator, that this semigroup may be extended to an analytic semigroup on a sector of complex plane. The results from the first section of this chapter, apart from Theorem 3.7, come from [27].

3.1. Case with mixed boundary conditions

Applying the identity (2.15) and then making use of Definition 2.18, we note that

$$\frac{\partial}{\partial x}D^\alpha u = \frac{\partial}{\partial x}I^{1-\alpha}u_x = \partial^\alpha u_x, \tag{3.1}$$

whenever one of the sides of this identity is meaningful. By Proposition 2.32, the domain of ∂^α in $L^2(0,1)$ coincides with ${}_0H^\alpha(0,1)$. Thus, we may consider the domain of $\frac{\partial}{\partial x}D^\alpha$ as $\{u \in H^{1+\alpha}(0,1) : u_x \in {}_0H^\alpha(0,1)\}$. We complement the definition of domain with a boundary condition $u(1) = 0$ and we arrive at

$$D(\frac{\partial}{\partial x}D^\alpha) \equiv \mathcal{D}_\alpha := \{u \in H^{1+\alpha}(0,1) : u_x \in {}_0H^\alpha(0,1),\ \ u(1) = 0\}. \tag{3.2}$$

We equip $\mathcal{D}_\alpha$ with the norm

$$\|f\|_{\mathcal{D}_\alpha} = \|f\|_{H^{1+\alpha}(0,1)} \ \text{ for } \alpha \in (0,1)\setminus\{\frac{1}{2}\}$$

and

$$\|f\|_{\mathcal{D}_\alpha} = \left(\|f\|^2_{H^{\frac{3}{2}}(0,1)} + \int_0^1 \frac{|u_x(x)|^2}{x}dx\right)^{\frac{1}{2}} \ \text{ for } \alpha = \frac{1}{2}.$$

For clarity, let us describe how the space $\mathcal{D}_{\alpha}$ looks like in dependence of α. If $\alpha \in (0, \frac{1}{2})$ we have $\mathcal{D}_{\alpha} = \{u \in H^{1+\alpha}(0,1): \ u(1) = 0\}$, for $\alpha = \frac{1}{2}$ there holds $\mathcal{D}_{\alpha} = \{u \in H^{\frac{3}{2}}(0,1): u_x \in {}_0H^{\frac{1}{2}}(0,1), \ u(1) = 0\}$ and in the case $\alpha \in (\frac{1}{2}, 1)$ we have $\mathcal{D}_{\alpha} = \{u \in H^{1+\alpha}(0,1): u_x(0) = 0, \ u(1) = 0\}$.

We note that if we prove that $\frac{\partial}{\partial x}D^{\alpha} : \mathcal{D}_{\alpha} \to L^2(0,1)$ generates an analytic semigroup, we will obtain existence and regularity results for a solution to

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^{\alpha}u = f & \text{in } (0,1)\times(0,T), \\ u_x \in {}_0H^{\alpha}(0,1), \quad u(1,t) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1). \end{cases} \tag{3.3}$$

In forthcoming sections we will also discuss the case with different boundary conditions. However, at first we investigate (3.3).

Theorem 3.1. *Operator $\frac{\partial}{\partial x}D^{\alpha} : \mathcal{D}_{\alpha} \subseteq L^2(0,1) \to L^2(0,1)$ generates a C_0 semigroup of contractions.*

Proof. We will prove Theorem 3.1 applying the Lumer-Philips theorem (see Theorem 2.9). At first we note that $\frac{\partial}{\partial x}D^{\alpha}$ is densely defined in $L^2(0,1)$, because $C_0^{\infty}(0,1) \subseteq \mathcal{D}_{\alpha}$. In order to verify the assumptions of Lumer-Philips theorem we need to show in addition that $\frac{\partial}{\partial x}D^{\alpha}$ is dissipative and that $R(I - \frac{\partial}{\partial x}D^{\alpha}) = L^2(0,1)$. (We recall that the definition of dissipative operator was given in Definition 2.12). In order to show dissipativity of $\frac{\partial}{\partial x}D^{\alpha}$ we consider $u \in \mathcal{D}_{\alpha}$. Since $u_x \in {}_0H^{\alpha}(0,1)$, then from Corollary 2.33 we know that $D^{\alpha}u = I^{1-\alpha}u_x \in {}_0H^1(0,1)$. Hence, in particular $D^{\alpha}u \in AC[0,1]$ and $(D^{\alpha}u)(0) = 0$. We apply the integration by parts formula and Proposition 2.30 to obtain

$$\operatorname{Re}\left(-\frac{\partial}{\partial x}D^{\alpha}u, u\right) = -\operatorname{Re}\int_0^1 (\frac{\partial}{\partial x}D^{\alpha}u)(x)\cdot\overline{u(x)}dx$$

$$= \int_0^1 D^{\alpha}\operatorname{Re}u(x)\cdot\frac{\partial}{\partial x}\operatorname{Re}u(x)dx + \int_0^1 D^{\alpha}\operatorname{Im}u(x)\cdot\frac{\partial}{\partial x}\operatorname{Im}u(x)dx$$

$$= \int_0^1 D^{\alpha}\operatorname{Re}u(x)\cdot\partial^{1-\alpha}D^{\alpha}\operatorname{Re}u(x)dx + \int_0^1 D^{\alpha}\operatorname{Im}u(x)\cdot\partial^{1-\alpha}D^{\alpha}\operatorname{Im}u(x)dx.$$

We may apply inequality (2.19) with $w = D^{\alpha}\operatorname{Re}u$ and $w = D^{\alpha}\operatorname{Im}u$ to obtain

$$\operatorname{Re}\left(-\frac{\partial}{\partial x}D^{\alpha}u, u\right) \geq c_{\alpha}\left\|D^{\alpha}u\right\|^2_{H^{\frac{1-\alpha}{2}}(0,1)} \geq c_{\alpha}\left\|\partial^{\frac{1-\alpha}{2}}D^{\alpha}u\right\|^2_{L^2(0,1)}$$

$$= c_{\alpha}\left\|D^{\frac{1+\alpha}{2}}u\right\|^2_{L^2(0,1)}, \tag{3.4}$$

where in the second inequality we used Proposition 2.32 together with the fact that $\frac{1-\alpha}{2} < \frac{1}{2}$ and the equality follows from Proposition 2.30. Here $c_{\alpha} > 0$ denotes a generic constant dependent on α.

Now, we would like to show that $R(E - \frac{\partial}{\partial x}D^{\alpha}) = L^2(0,1)$. In fact, we are able to show something more. We will state the result in the next lemma.

Lemma 3.2. *For every $\lambda \in \mathbb{C}$ belonging to the sector*

$$\vartheta_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \le \frac{\pi(\alpha+1)}{2}\} \cup \{0\} \tag{3.5}$$

there holds

$$R(\lambda E - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1).$$

Proof. To prove the lemma we fix $g \in L^2(0,1)$ and λ belonging to ϑ_α. We must prove that there exists $u \in \mathcal{D}_\alpha$ such that

$$\lambda u - \frac{\partial}{\partial x} D^\alpha u = g. \tag{3.6}$$

We would like to calculate the solution directly. To that end, we will firstly solve equation (3.6) with an arbitrary boundary condition $u(0) = u_0 \in \mathbb{C}$. Then, we will choose u_0 which will guarantee the zero condition at the other endpoint of the interval. We note that if we search for a solution in $\{f \in H^{1+\alpha}(0,1) : f_x \in {}_0H^\alpha(0,1)\}$, then equation (3.6) is equivalent to

$$u = u_0 + \lambda I^{\alpha+1} u - I^{\alpha+1} g. \tag{3.7}$$

Indeed, if we recall identity (3.1) and assume that $u_x \in {}_0H^\alpha(0,1)$, then applying I^α to both sides of (3.6) yields

$$u_x = \lambda I^\alpha u - I^\alpha g.$$

After having integrated this equality we arrive at (3.7). On the other hand, if we assume that $u \in L^2(0,1)$ solves (3.7), then by Proposition 2.32 it automatically belongs to $\{f \in H^{1+\alpha}(0,1) : f_x \in {}_0H^\alpha(0,1)\}$ and in order to obtain (3.6) it is enough to apply $\partial^\alpha \frac{\partial}{\partial x}$ to (3.7).

Thus, we are going to solve (3.7). For this purpose, we apply to (3.7) the operator $I^{\alpha+1}$ and we obtain

$$I^{\alpha+1} u = I^{\alpha+1} u_0 + \lambda I^{2(\alpha+1)} u - I^{2(\alpha+1)} g.$$

Inserting this result in (3.7) we get

$$u(x) = u_0 - (I^{\alpha+1} g)(x) + \lambda (I^{\alpha+1} u_0)(x) + \lambda^2 (I^{2(\alpha+1)} u)(x) - \lambda (I^{2(\alpha+1)} g)(x).$$

Iterating this procedure n times we arrive at

$$u(x) = u_0 \sum_{k=0}^{n} (\lambda^k I^{k(\alpha+1)} 1)(x) - \sum_{k=0}^{n} \lambda^k (I^{(k+1)(\alpha+1)} g)(x) + \lambda^{n+1} (I^{(n+1)(\alpha+1)} u)(x). \tag{3.8}$$

We will show, that the last expression tends to zero as $n \to \infty$. Indeed, we may note that, since we search for the solutions in $H^{1+\alpha}(0,1) \subseteq L^\infty(0,1)$ and due to the presence of the Γ-function in the denominator we have

$$\left|\lambda^n (I^{n(\alpha+1)} u)(x)\right| \le \|u\|_{L^\infty(0,1)} \frac{|\lambda|^n x^{(\alpha+1)n}}{\Gamma((\alpha+1)n+1)} \le \frac{\|u\|_{L^\infty(0,1)} |\lambda|^n}{\Gamma((\alpha+1)n+1)} \to 0 \text{ as } n \to \infty$$

for each $\lambda \in \mathbb{C}$ uniformly with respect to $x \in [0,1]$. Thus, passing to the limit with n in (3.8) we obtain the formula

$$u(x) = u_0 \sum_{k=0}^{\infty} (\lambda^k I^{k(\alpha+1)} 1)(x) - \sum_{k=0}^{\infty} \lambda^k (I^{(k+1)(\alpha+1)} g)(x). \tag{3.9}$$

We will show that both series in (3.9) are uniformly convergent and we will calculate their sums. Indeed, we may directly compute the sum of the series. We note that by the Example 2.2

$$(I^{k(\alpha+1)}1)(x) = \frac{x^{k(\alpha+1)}}{\Gamma(k(\alpha+1)+1)}.$$

Hence,

$$\sum_{k=0}^{\infty} \lambda^k (I^{(\alpha+1)k}1)(x) = E_{\alpha+1}(\lambda x^{\alpha+1}), \tag{3.10}$$

where $E_{\alpha+1}$ denotes the Mittag-Leffler function given by Definition 2.19. To calculate the sum of the second series, we apply the definition of fractional integral (see Definition 2.18)

$$I^{(\alpha+1)(k+1)}g(x) = \int_0^x g(s)\frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k+\alpha+1)}ds.$$

In order to interchange the order of integration and summation, we will firstly consider the finite sum and then we will pass to the limit,

$$\sum_{k=0}^{\infty} \lambda^k \int_0^x g(s)\frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k+\alpha+1)}ds = \lim_{n\to\infty}\sum_{k=0}^{n} \lambda^k \int_0^x g(s)\frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k+\alpha+1)}ds$$

$$= \lim_{n\to\infty}\int_0^x g(s)\sum_{k=0}^{n} \lambda^k \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k+\alpha+1)}ds.$$

We would like to apply the Lebesgue dominated convergence theorem, thus we need to indicate the majorant. We may estimate as follows

$$\left|g(s)\sum_{k=0}^{n} \lambda^k \frac{(x-s)^{(\alpha+1)k+\alpha}}{\Gamma((\alpha+1)k+\alpha+1)}\right| \le |g(s)|\sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma((\alpha+1)k+\alpha+1)}$$

$$= |g(s)|\, E_{\alpha+1,\alpha+1}(|\lambda|)$$

and the last function is integrable because $g \in L^2(0,1)$. Hence, applying the Lebesgue dominated convergence theorem we arrive at

$$\sum_{k=0}^{\infty} \lambda^k (I^{(\alpha+1)(k+1)}g)(x) = g * x^{\alpha}\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha+1})^k}{\Gamma((\alpha+1)k+(\alpha+1))}.$$

Finally, using this result together with (3.10) we obtain that the function u given by (3.9) may be written by the following formula

$$u(x) = u_0 E_{\alpha+1}(\lambda x^{\alpha+1}) - g * x^{\alpha} E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}). \tag{3.11}$$

We note that this function actually satisfies (3.7). Indeed, applying Example 2.2 we may calculate

$$\lambda I^{\alpha+1}u = u_0\sum_{k=0}^{\infty}\lambda^{k+1}I^{(\alpha+1)(k+1)}1(x) - \lambda\sum_{k=0}^{\infty}I^{(\alpha+1)(k+2)}g(x)$$

$$= u_0\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha+1})^{k+1}}{\Gamma((\alpha+1)(k+1)+1)} - g * \sum_{k=0}^{\infty}\frac{\lambda^{k+1}x^{(\alpha+1)(k+2)-1}}{\Gamma((\alpha+1)(k+2))}$$

$$= u_0\sum_{k=1}^{\infty}\frac{(\lambda x^{\alpha+1})^{k}}{\Gamma((\alpha+1)k+1)} - g * \sum_{k=1}^{\infty}\frac{\lambda^{k}x^{(\alpha+1)(k+1)-1}}{\Gamma((\alpha+1)(k+1))}.$$

Hence,

$$u_0+\lambda I^{\alpha+1}u-I^{\alpha+1}g = u_0\left[1+\sum_{k=1}^{\infty}\frac{(\lambda x^{\alpha+1})^k}{\Gamma((\alpha+1)k+1)}\right]-g*\left[\sum_{k=1}^{\infty}\frac{\lambda^k x^{(\alpha+1)(k+1)-1}}{\Gamma((\alpha+1)(k+1))}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right]$$

and the last formula is equal to u. Thus, u given by formula (3.11) is a solution to (3.7) and hence it is also a solution to (3.6) with a boundary condition $u(0)=u_0$. It remains to solve equation (3.6) with the zero condition at the right endpoint of the interval. For this purpose, we take $x=1$ in (3.11) and we obtain

$$u(1)=u_0E_{\alpha+1}(\lambda)-(g*y^{\alpha}E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1).$$

To obtain $u(1)=0$ we take

$$u_0=(E_{\alpha+1}(\lambda))^{-1}(g*y^{\alpha}E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1).$$

We note that u_0 is well defined because, taking $\nu=\alpha+1, \mu=1$ in Proposition 2.27, we obtain that $E_{\alpha+1}(\lambda)\neq 0$ for λ belonging to the sector ϑ. Inserting this u_0 in (3.11) we obtain a formula for a solution to (3.6) which belongs to $\mathcal{D}_{\alpha}$:

$$u(x)=(E_{\alpha+1}(\lambda))^{-1}(g*y^{\alpha}E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1)E_{\alpha+1}(\lambda x^{\alpha+1})-g*x^{\alpha}E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

In this way we proved the lemma. □

Lemma 3.2 together with the dissipative property of $\frac{\partial}{\partial x}D^{\alpha}$ allows us to apply Theorem 2.9, which finishes the proof of Theorem 3.1. □

Our next goal is to prove that the semigroup generated by $\frac{\partial}{\partial x}D^{\alpha}$ can be extended to an analytic semigroup on a sector of complex plane. Before we will prove that result, we need to formulate two auxiliary lemmas. A similar reasoning to the one carried in Lemma 3.3 may be found in [10, Lemma 6].

Lemma 3.3. *The formulas* $\left\|D^{\frac{1+\alpha}{2}}u\right\|_{L^2(0,1)}$ *and* $\|u\|_{H^{\frac{1+\alpha}{2}}(0,1)}$ *define equivalent norms on* $\mathcal{D}_{\alpha}$.

Proof. We denote by c_{α} a generic constant dependent only on α. Firstly, we will show that there exists c_{α} such that

$$\left\|D^{\frac{1+\alpha}{2}}u\right\|_{L^2(0,1)}\leq c_{\alpha}\|u\|_{H^{\frac{1+\alpha}{2}}(0,1)}.$$

Using formula (2.15) and Proposition 2.35 we may write

$$\left\|D^{\frac{1+\alpha}{2}}u\right\|_{L^2(0,1)}=\left\|I^{\frac{1-\alpha}{2}}u_x\right\|_{L^2(0,1)}\leq c_{\alpha}\|u_x\|_{H^{\frac{\alpha-1}{2}}(0,1)}.$$

Due to Remark 2.2 we know that $\frac{\partial}{\partial x}$ is a bounded and linear operator from $H^s(0,1)$ to $H^{s-1}(0,1)$ for $s\in[0,1]\setminus\{\frac{1}{2}\}$ thus

$$\left\|D^{\frac{1+\alpha}{2}}u\right\|_{L^2(0,1)}\leq c_{\alpha}\|u\|_{H^{\frac{\alpha+1}{2}}(0,1)}.$$

To show the opposite inequality we notice that since $u\in\mathcal{D}_{\alpha}$ we have

$$u(x)=-\int_x^1 u_x(s)ds=-I_-^{\frac{1+\alpha}{2}}I_-^{\frac{1-\alpha}{2}}u_x(x),$$

where we applied Proposition 2.22 to right-side fractional integral defined in Definition 2.20. Thus, by Proposition 2.32 we may estimate

$$\|u\|_{H^{\frac{\alpha+1}{2}}(0,1)} = \left\| I_{-}^{\frac{1+\alpha}{2}} I_{-}^{\frac{1-\alpha}{2}} u_x \right\|_{{}^0H^{\frac{\alpha+1}{2}}(0,1)} \le c_\alpha \left\| I_{-}^{\frac{1-\alpha}{2}} u_x \right\|_{L^2(0,1)}.$$

Applying Proposition 2.35 and Proposition 2.36 we may estimate further

$$\begin{aligned} \|u\|_{H^{\frac{\alpha+1}{2}}(0,1)} &\le c_\alpha \|u_x\|_{H^{\frac{\alpha-1}{2}}(0,1)} = c_\alpha \left\| \partial^{\frac{1-\alpha}{2}} I^{\frac{1-\alpha}{2}} u_x \right\|_{H^{\frac{\alpha-1}{2}}(0,1)} \\ &\le c_\alpha \left\| I^{\frac{1-\alpha}{2}} u_x \right\|_{L^2(0,1)} = c_\alpha \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)}, \end{aligned}$$

which finishes the proof. □

Lemma 3.4. *For $u \in \mathcal{D}_\alpha$ we have*

$$\operatorname{Re}(-\frac{\partial}{\partial x} D^\alpha u, u) \ge c_\alpha \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)} \tag{3.12}$$

and

$$\left|(-\frac{\partial}{\partial x} D^\alpha u, u)\right| \le b_\alpha \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}, \tag{3.13}$$

where c_α, b_α are positive constant which depends only on α.

Proof. We have already obtained in (3.4) that

$$\operatorname{Re}\left(-\frac{\partial}{\partial x} D^\alpha u, u\right) \ge c_\alpha \left\| D^{\frac{1+\alpha}{2}} u \right\|^2_{L^2(0,1)}.$$

Hence, in order to prove (3.12) it is enough to apply the norm equivalence from Lemma 3.3. Now, we will prove (3.13). In fact, we will show something more, i. e. there exists $b_\alpha > 0$ such that for every $u \in \mathcal{D}_\alpha$ and every $w \in AC[0,1] \cap {}^0H^{\frac{1+\alpha}{2}}(0,1)$ there holds

$$\left|(-\frac{\partial}{\partial x} D^\alpha u, w)\right| \le b_\alpha \|u\|_{H^{\frac{1+\alpha}{2}}(0,1)} \|w\|_{{}^0H^{\frac{1+\alpha}{2}}(0,1)}. \tag{3.14}$$

At first, we notice that since $u \in \mathcal{D}_\alpha$, we know that $u_x \in {}_0H^\alpha(0,1)$ and from Corollary 2.33 we infer that $D^\alpha u = I^{1-\alpha} u_x \in {}_0H^1(0,1)$. Applying Remark 2.7 in the first identity below and Proposition 2.30 in the second one, we may write

$$\frac{\partial}{\partial x} D^\alpha u = \partial^{\frac{1+\alpha}{2}} \partial^{\frac{1-\alpha}{2}} D^\alpha u = \partial^{\frac{1+\alpha}{2}} D^{\frac{1+\alpha}{2}} u = \frac{\partial}{\partial x} I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u.$$

We integrate by parts and make use of $\overline{w}(1) = 0$, $(D^\alpha u)(0) = 0$, the identity (2.23) and Definition 2.20 to get

$$(\frac{\partial}{\partial x} D^\alpha u, w) = \int_0^1 \frac{\partial}{\partial x} I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u \cdot \overline{w} dx = -\int_0^1 I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u \cdot \overline{w}_x dx = \int_0^1 D^{\frac{1+\alpha}{2}} u \cdot D_{-}^{\frac{1+\alpha}{2}} \overline{w} dx. \tag{3.15}$$

Thus,

$$\left|(-\frac{\partial}{\partial x} D^\alpha u, w)\right| = \left| \int_0^1 D^{\frac{1+\alpha}{2}} u \cdot D_{-}^{\frac{1+\alpha}{2}} \overline{w} dx \right| \le \left\| D^{\frac{1+\alpha}{2}} u \right\|_{L^2(0,1)} \left\| D_{-}^{\frac{1+\alpha}{2}} w \right\|_{L^2(0,1)}. \tag{3.16}$$

Since $w(1) = 0$, applying Proposition 2.32 we obtain that

$$\left\| D_{-}^{\frac{1+\alpha}{2}} w \right\|_{L^2(0,1)} = \left\| \partial_{-}^{\frac{1+\alpha}{2}} w \right\|_{L^2(0,1)} \le b_\alpha \|w\|_{{}^0H^{\frac{1+\alpha}{2}}(0,1)} = b_\alpha \|w\|_{H^{\frac{1+\alpha}{2}}(0,1)},$$

where by b_α we denote a positive constant dependent on α. Making use of this estimate and the norm equivalence from Lemma 3.3 in (3.16) we obtain (3.14). Putting $w = u$ we arrive at estimate (3.13). □

Finally, we are ready to prove the main theorem.

Theorem 3.5. *The operator* $\frac{\partial}{\partial x}D^\alpha : \mathcal{D}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ *is densely defined sectorial operator, thus it generates an analytic semigroup.*

Proof. We will give the proof of analyticity following the proof of [23, Ch. 7, Theorem 2.7], where the elliptic operators are studied.
At first, we notice that since $L^2(0,1)$ is a Hilbert space, the numerical range of $-\frac{\partial}{\partial x}D^\alpha$ (see Proposition 2.10) equals

$$S(-\frac{\partial}{\partial x}D^\alpha) = \left\{ \left(u, -\frac{\partial}{\partial x}D^\alpha u\right) : u \in \mathcal{D}_\alpha,\ \ \|u\|_{L^2(0,1)} = 1 \right\}.$$

Indeed, let us assume that there exists $w \in L^2(0,1)$ such that $\|w\|_{L^2(0,1)} = 1$ and $(w,u) = 1$ for u such that $\|u\|_{L^2(0,1)} = 1$. Then

$$\|w-u\|^2_{L^2(0,1)} = \|w\|^2_{L^2(0,1)} - 2\operatorname{Re}(w,u) + \|u\|^2_{L^2(0,1)} = 0,$$

hence $w = u$. We note that by (3.12) zero does not belong to $S(-\frac{\partial}{\partial x}D^\alpha)$. Let us denote $z = \left(u, -\frac{\partial}{\partial x}D^\alpha u\right)$. Then, in view of (3.12) and (3.13), we obtain that

$$|\tan(\arg z)| = \left|\frac{\operatorname{Im} z}{\operatorname{Re} z}\right| \le \frac{b_\alpha}{c_\alpha},$$

which implies

$$S(-\frac{\partial}{\partial x}D^\alpha) \subseteq \left\{ \lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| \le \arctan\left(\frac{b_\alpha}{c_\alpha}\right) \right\}$$

and $\arctan(\frac{b_\alpha}{c_\alpha}) < \frac{\pi}{2}$. We may choose ν such that $\arctan(\frac{b_\alpha}{c_\alpha}) < \nu < \frac{\pi}{2}$ and denote $\Sigma_\nu := \{\lambda : \lambda \neq 0,\ \ |\arg\lambda| > \nu\}$. Then, $\Sigma_\nu \subseteq \mathbb{C} \setminus \overline{S(-\frac{\partial}{\partial x}D^\alpha)}$. We will show that there exists a positive constant c_ν such that

$$d(\lambda, S(-\frac{\partial}{\partial x}D^\alpha)) \ge c_\nu\,|\lambda| \ \text{ for all } \lambda \in \Sigma_\nu. \tag{3.17}$$

Indeed, in the case when $\lambda \in \Sigma_\nu$ is such that $|\arg\lambda| > \frac{\pi}{2} + \arctan(\frac{b_\alpha}{c_\alpha})$ we obtain that $d(\lambda, S(-\frac{\partial}{\partial x}D^\alpha)) \ge |\lambda|$. If we assume that $\nu < \arg\lambda \le \frac{\pi}{2} + \arctan(\frac{b_\alpha}{c_\alpha})$ we arrive at

$$\frac{d(\lambda, S(-\frac{\partial}{\partial x}D^\alpha))}{|\lambda|} \ge \sin(\arg\lambda - \arctan(\frac{b_\alpha}{c_\alpha})) \ge \sin(\nu - \arctan(\frac{b_\alpha}{c_\alpha})).$$

Finally, if $-\frac{\pi}{2} - \arctan(\frac{b_\alpha}{c_\alpha}) \le \arg\lambda < -\nu$ we get that

$$\frac{d(\lambda, S(-\frac{\partial}{\partial x}D^\alpha))}{|\lambda|} \ge \left|\sin(\arg\lambda + \arctan(\frac{b_\alpha}{c_\alpha}))\right| \ge \left|\sin(-\nu + \arctan(\frac{b_\alpha}{c_\alpha}))\right|$$

and we obtain (3.17). By Theorem 3.1 we know that $(-\infty, 0] \subseteq \rho(-\frac{\partial}{\partial x}D^\alpha)$, which implies that

$$\Sigma_\nu \cap \rho(-\frac{\partial}{\partial x}D^\alpha) \neq \emptyset.$$

We may apply Proposition 2.10 to the operator $-\frac{\partial}{\partial x}D^\alpha$ to obtain that spectrum of $-\frac{\partial}{\partial x}D^\alpha$ is contained in $\mathbb{C}\setminus\Sigma_\nu$, which means that $\Sigma_\nu \subseteq \rho(-\frac{\partial}{\partial x}D^\alpha)$ and

$$\left\|\left(\lambda E - (-\frac{\partial}{\partial x}D^\alpha)\right)^{-1}\right\| \le \frac{1}{d(\lambda, \overline{S(\frac{\partial}{\partial x}D^\alpha)})} \le \frac{1}{c_\nu\,|\lambda|} \ \text{for all } \lambda \in \Sigma_\nu.$$

Thus, the set $\{\lambda \in \mathbb{C} : |\arg \lambda| < \pi - \nu\} \cup \{0\} \subseteq \rho(\frac{\partial}{\partial x}D^\alpha)$ and

$$\left\| \left(\lambda E - \frac{\partial}{\partial x}D^\alpha\right)^{-1} \right\| \leq \frac{1}{c_\nu |\lambda|} \text{ for every } \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \pi - \nu.$$

Hence, we showed that $\frac{\partial}{\partial x}D^\alpha$ is sectorial and the semigroup generated by $\frac{\partial}{\partial x}D^\alpha$ can be extended to the analytic semigroup on a sector of complex plane. □

We finish this section with a simple application of obtained results.

Theorem 3.6. *Let us consider problem (3.3) with $f \equiv 0$. If we assume that $u_0 \in L^2(0,1)$, then there exists exactly one solution to (3.3) which belongs to $C([0,T]; L^2(0,1)) \cap C((0,T]; \mathcal{D}_\alpha) \cap C^1((0,T]; L^2(0,1))$. Furthermore, there exists a positive constant $c = c(T)$, such that the following estimate holds for every $t \in (0,T]$*

$$\|u(\cdot,t)\|_{L^2(0,1)} + t\,\|u_t(\cdot,t)\|_{L^2(0,1)} + t \left\| \frac{\partial}{\partial x}D^\alpha u(\cdot,t) \right\|_{L^2(0,1)} \leq c\,\|u_0\|_{L^2(0,1)}.$$

Nevertheless, $u \in C^\infty((0,T]; L^2(0,1))$ and for every $t \in (0,T]$, for very $k \in \mathbb{N}$ we have $u(\cdot,t) \in D((\frac{\partial}{\partial x}D^\alpha)^k)$. The last property implies that $u(\cdot,t) \in C^\infty(0,1)$ for every $t \in (0,T]$, however u has a singularity of the form $x^{\alpha+1}$ at the left endpoint of the interval.

Proof. Since, we know that the operator $\frac{\partial}{\partial x}D^\alpha$ generates an analytic semigroup, we may apply to (3.3) the general semigroup theory. Then, in view of Theorem 2.11, it remains to describe how a domain of k -th power of $\frac{\partial}{\partial x}D^\alpha$ looks like. Let us focus on $k = 2$. Then, $u \in D((\frac{\partial}{\partial x}D^\alpha)^2)$ if $u \in \mathcal{D}_\alpha$ and $\frac{\partial}{\partial x}D^\alpha u \in \mathcal{D}_\alpha$. Applying Proposition 2.32 we obtain that

$$u_x = I^\alpha \partial^\alpha u_x = I^\alpha(\partial^\alpha u_x - (\partial^\alpha u_x)(0)) + (\partial^\alpha u_x)(0) I^\alpha 1.$$

We integrate this identity and apply Proposition 2.22 and Example 2.2

$$u(x) = u(0) + I^{1+\alpha}(\partial^\alpha u_x - (\partial^\alpha u_x)(0)) + (\partial^\alpha u_x)(0)\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}.$$

By the assumption and identity (3.1) there holds $\partial^\alpha u_x - (\partial^\alpha u_x)(0) \in {}_0H^{1+\alpha}(0,1)$. Hence, by Corollary 2.33 we obtain that $I^{1+\alpha}(\partial^\alpha u_x - (\partial^\alpha u_x)(0))$ belongs to ${}_0H^{2(1+\alpha)}(0,1)$. In the case $u \in D((\frac{\partial}{\partial x}D^\alpha)^k)$ we iterate the above procedure and we arrive at

$$u(x) = \sum_{n=0}^{k}((\frac{\partial}{\partial x}D^\alpha)^n u)(0)\frac{x^{n(1+\alpha)}}{\Gamma(1+n(1+\alpha))} + I^{k(1+\alpha)}((\frac{\partial}{\partial x}D^\alpha)^k u - ((\frac{\partial}{\partial x}D^\alpha)^k u)(0)).$$

By Corollary 2.33 the last component belongs to ${}_0H^{(1+\alpha)(k+1)}(0,1)$ and we note that function u has a singularity of the form $x^{\alpha+1}$ at the origin. By Theorem 2.11 a solution to (3.3) belongs to $\bigcap_{k=1}^\infty D((\frac{\partial}{\partial x}D^\alpha)^k)$, thus we obtained the claim. □

Remark 3.1. *One may consider the problem (3.3) with nonzero right-hand-side. Then the solution is obtained by the variation of constant formula, see for example Theorem 2.12.*

Remark 3.2. *The result of Theorem 3.5 may be extended to the case of operator $\frac{\partial}{\partial x}p(x)D^\alpha$: $\mathcal{D}_\alpha \to L^2(0,1)$, where $p \in W^{1,\infty}(0,1)$ is positive and separated away from zero. The proof may be found in [27], however, we skip the proof here, since it is technical.*

Let us show another possible generalization of this problem.

Theorem 3.7. *Let us consider the operator* $A : \mathcal{D}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ *defined by*

$$Au = \frac{\partial}{\partial x} D^\alpha u + \int_0^\beta \mu(\gamma) \frac{\partial}{\partial x} D^\gamma u d\gamma,$$

where $0 < \beta < \alpha < 1$ *and*

$$\mu(\gamma) = \sum_{k=1}^{M} q_k \delta(\cdot - \gamma_k) + \omega(\gamma).$$

We assume that $M \in \mathbb{N}$, $q_k \geq 0$ *for* $k = 1, \ldots, M$, $\gamma_k \in (0, \beta]$ *for* $k = 1, \ldots, M$ *and* $\omega \in L^1(0,\beta)$, $\omega \geq 0$. *Then,* $A : \mathcal{D}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ *is sectorial, hence it generates an analytic semigroup.*

Proof. In order to prove the theorem we will apply Proposition 2.14. Let us consider the operator $B : \mathcal{D}_\beta \to L^2(0,1)$, defined by $Bu := \int_0^\beta \mu(\gamma) \frac{\partial}{\partial x} D^\gamma u d\gamma$. We recall that the definition of space $\mathcal{D}_\beta$ was given in (3.2). Let us justify the definition of operator B. At first, we will show that for $0 < \gamma < \beta$ and $u \in \mathcal{D}_\beta$ function $\gamma \mapsto \mu(\gamma) \frac{\partial}{\partial x} D^\gamma u$ is measurable with values in $L^2(0,1)$. Indeed, applying identity (3.1) and Theorem 2.7 we obtain that for $u \in \mathcal{D}_\beta$ there holds $\frac{\partial}{\partial x} D^\gamma u = \partial^\gamma u_x = I^{\beta-\gamma} \partial^\beta u_x$. Applying Proposition 2.3 together with Definition 2.6 we obtain that the function $\gamma \mapsto I^{\beta-\gamma}\partial^\beta u_x$ is continuous on $[0,\beta]$ with values in $L^2(0,1)$. Thus, for $0 < \gamma < \beta$ and $u \in \mathcal{D}_\beta$ function $\gamma \mapsto \mu(\gamma) \frac{\partial}{\partial x} D^\gamma u$ is measurable with values in $L^2(0,1)$. We note that for $u \in \mathcal{D}_\beta$

$$\|Bu\|_{L^2(0,1)} \leq \int_0^\beta \mu(\gamma) \left\| \frac{\partial}{\partial x} D^\gamma u \right\|_{L^2(0,1)} d\gamma = \int_0^\beta \mu(\gamma) \left\| \partial^\gamma u_x \right\|_{L^2(0,1)} d\gamma.$$

Applying Proposition 2.4 we obtain that there exists $c > 0$ (independent on γ and β) such that for every $\gamma \in (0,\beta)$

$$\|\partial^\gamma u_x\|_{L^2(0,1)} \leq c \left\|\partial^\beta u_x\right\|_{L^2(0,1)}^{\frac{\gamma}{\beta}} \|u_x\|_{L^2(0,1)}^{\frac{\beta-\gamma}{\beta}}.$$

Hence, we may estimate further

$$\|\partial^\gamma u_x\|_{L^2(0,1)} \leq c(\beta) \|u\|_{\mathcal{D}_\beta}^{\frac{\gamma}{\beta}} \|u\|_{\mathcal{D}_\beta}^{\frac{\beta-\gamma}{\beta}} = c(\beta) \|u\|_{\mathcal{D}_\beta}$$

and

$$\|Bu\|_{L^2(0,1)} \leq c(\beta) \|u\|_{\mathcal{D}_\beta} \int_0^\beta \mu(\gamma) d\gamma = c(\beta,\mu) \|u\|_{\mathcal{D}_\beta}.$$

Thus, $B \in B(\mathcal{D}_\beta, L^2(0,1))$. We note that in our case we may show more direct estimate without the use of Proposition 2.4. Indeed, recalling that for every $f \in L^2(0,1)$ and every $\alpha > 0$ there holds $\|I^\alpha f\|_{L^2(0,1)} \leq \frac{1}{\Gamma(\alpha+1)} \|f\|_{L^2(0,1)}$ we may write

$$\|\partial^\gamma u_x\|_{L^2(0,1)} = \left\|I^{\beta-\gamma}\partial^\beta u_x\right\|_{L^2(0,1)} \leq \frac{1}{\Gamma(1+\beta-\gamma)} \left\|\partial^\beta u_x\right\|_{L^2(0,1)} \leq 2 \|u\|_{\mathcal{D}_\beta},$$

where in the last inequality we used the fact that $\Gamma(\cdot) > \frac{1}{2}$ on $[1,2]$.
Furthermore, $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta \subseteq L^2(0,1)$ and for every $u \in \mathcal{D}_\alpha$ we have

$$\|u\|_{\mathcal{D}_\beta} \leq c \|u_x\|_{{}_0H^\beta(0,1)} \leq c(\alpha,\beta) \|u_x\|_{{}_0H^\alpha(0,1)}^{\frac{\beta}{\alpha}} \|u_x\|_{L^2(0,1)}^{1-\frac{\beta}{\alpha}} \leq c(\alpha,\beta) \|u\|_{\mathcal{D}_\alpha}^{\frac{\beta}{\alpha}} \|u_x\|_{L^2(0,1)}^{1-\frac{\beta}{\alpha}},$$

where in the first estimate we applied the Poincaré inequality, while in the second one we applied interpolation estimate ([19, Corollary 1.2.7.]). Applying again ([19, Corollary 1.2.7.]) we may write

$$\|u_x\|_{L^2(0,1)} \leq \|u\|_{H^1(0,1)} \leq c(\alpha)\,\|u\|_{L^2(0,1)}^{\frac{\alpha}{\alpha+1}}\,\|u\|_{H^{1+\alpha}(0,1)}^{\frac{1}{\alpha+1}} \leq c(\alpha)\,\|u\|_{L^2(0,1)}^{\frac{\alpha}{\alpha+1}}\,\|u\|_{\mathcal{D}_\alpha}^{\frac{1}{\alpha+1}}.$$

Together we obtain that

$$\|u\|_{\mathcal{D}_\beta} \leq c(\alpha,\beta)\,\|u\|_{L^2(0,1)}^{\frac{\alpha-\beta}{\alpha+1}}\,\|u\|_{\mathcal{D}_\alpha}^{\frac{1+\beta}{\alpha+1}}.$$

Hence, the claim follows from Proposition 2.14.

□

3.2. Case with Dirichlet boundary conditions

In this section we will consider the problem with Dirichlet boundary conditions.

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^\alpha u = f & \text{in } (0,1)\times(0,T), \\ u(0,t) = 0, \quad u(1,t) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{3.18}$$

To this end, we have to redefine the domain of $\frac{\partial}{\partial x}D^\alpha$. Let us introduce

$$\bar{\mathcal{D}_\alpha} := \{u = w - w(1)x^\alpha, \text{ where } w \in {}_0H^{1+\alpha}(0,1)\}.$$

From Example 2.3 we infer that $x^\alpha \notin {}_0H^{1+\alpha}(0,1)$. Thus, for every $u \in \bar{\mathcal{D}_\alpha}$ the function w is uniquely determined. Indeed, if we assume that there exist $w_1, w_2 \in {}_0H^{1+\alpha}(0,1)$ such that $w_1 - w_1(1)x^\alpha = w_2 - w_2(1)x^\alpha$, then $w_1(1) = w_2(1)$ and $w_1 = w_2$ in ${}_0H^{1+\alpha}(0,1)$. We equip $\bar{\mathcal{D}_\alpha}$ with the following norm

$$\|u\|_{\bar{\mathcal{D}_\alpha}} = \|w\|_{H^{1+\alpha}(0,1)} \text{ for } \alpha \in (0,1)\setminus\{\frac{1}{2}\}$$

and

$$\|u\|_{\bar{\mathcal{D}_\alpha}} = \left(\|w\|^2_{H^{\frac{3}{2}}(0,1)} + \int_0^1 \frac{|w_x(x)|^2}{x}dx\right)^{\frac{1}{2}} \text{ for } \alpha = \frac{1}{2}.$$

By identity (3.1) and Example 2.3 we may easily calculate $\frac{\partial}{\partial x}D^\alpha x^\alpha = 0$. Thus,

$$\frac{\partial}{\partial x}D^\alpha : \bar{\mathcal{D}_\alpha} \to L^2(0,1).$$

Moreover $\bar{\mathcal{D}_\alpha}$ is dense in $L^2(0,1)$, because $C_0^\infty(0,1) \subseteq \bar{\mathcal{D}_\alpha}$. We will show that $\frac{\partial}{\partial x}D^\alpha$ defined on $\bar{\mathcal{D}_\alpha}$ is also a generator of analytic semigroup on $L^2(0,1)$. The strategy of the proof is as follows. We will prove Lemma 3.2 and Lemma 3.4 for $\frac{\partial}{\partial x}D^\alpha$ defined on $\bar{\mathcal{D}_\alpha}$ and then we will repeat the proof of Theorem 3.5 to obtain the claim. We will begin with the analysis of the resolvent.

Lemma 3.8. *Let us discuss $\frac{\partial}{\partial x}D^\alpha : \bar{\mathcal{D}_\alpha} \to L^2(0,1)$. Then, for every $\lambda \in \mathbb{C}$ belonging to the sector*

$$\vartheta_\alpha := \{z \in \mathbb{C}\setminus\{0\} : |\arg z| \leq \frac{\pi(\alpha+1)}{2}\} \cup \{0\} \tag{3.19}$$

there holds

$$R(\lambda E - \frac{\partial}{\partial x}D^\alpha) = L^2(0,1).$$

Proof. To prove the lemma we fix $g \in L^2(0,1)$ and λ belonging to ϑ_α. We must prove that there exists $u \in \bar{\mathcal{D}}_\alpha$ such that

$$\lambda u - \frac{\partial}{\partial x}D^\alpha u = g. \tag{3.20}$$

We will proceed as in the proof of Lemma 3.2. We note that if we search for a solution in $\bar{\mathcal{D}}_\alpha$, then it can be represented in the form $u = w - w(1)x^\alpha$ and, by Example 2.1, $D^\alpha u = D^\alpha w - w(1)\Gamma(\alpha+1)$. Since $w \in {}_0H^{1+\alpha}(0,1)$, we have $D^\alpha w \in {}_0H^1(0,1)$ and hence $D^\alpha u(0) = -w(1)\Gamma(\alpha+1)$. At first we will solve the equation (3.20) with initial conditions $u(0) = 0$ and $D^\alpha u(0) = a$ for arbitrary $a \in \mathbb{C}$ and then we will choose a such that $u(1) = 0$. At first, let us transform the equation (3.20) into integral form. To this end we assume that u which may be written in the form $u = w + \frac{a}{\Gamma(1+\alpha)}x^\alpha$ where $w \in {}_0H^{1+\alpha}(0,1)$, solves (3.20). Then, having integrated (3.20) we obtain that

$$D^\alpha u = (D^\alpha u)(0) + \lambda I u - I g = a + \lambda I u - I g.$$

Applying I^α we get

$$u = aI^\alpha 1 + \lambda I^{\alpha+1}u - I^{\alpha+1}g. \tag{3.21}$$

We note that if we search for a solution such that there exists $w \in {}_0H^{1+\alpha}(0,1)$, such that $u = w + \frac{a}{\Gamma(1+\alpha)}x^\alpha$, then equation (3.20) is equivalent with (3.21). Indeed, it follows from Proposition 2.32 together with Example 2.2.

We apply the operator $I^{\alpha+1}$ to (3.21) and we obtain

$$u = aI^\alpha 1 - I^{\alpha+1}g + \lambda a I^{2\alpha+1}1 + \lambda^2 I^{2(\alpha+1)}u - \lambda I^{2(\alpha+1)}g.$$

Iterating this procedure n times we arrive at

$$u = a\sum_{k=0}^{n} \lambda^k I^{\alpha+k(\alpha+1)}1 - \sum_{k=0}^{n}\lambda^k I^{(k+1)(\alpha+1)}g + \lambda^{n+1}I^{(n+1)(\alpha+1)}u. \tag{3.22}$$

We will show, that the last expression tends to zero as $n \to \infty$. Indeed, we may note that, since ${}_0H^{1+\alpha}(0,1) \subseteq L^\infty(0,1)$ and due to the presence of the Γ-function in the denominator we have

$$\left|\lambda^n (I^{n(\alpha+1)}u)(x)\right| \le \|u\|_{L^\infty(0,1)} \frac{|\lambda|^n x^{(\alpha+1)n}}{\Gamma((\alpha+1)n+1)} \le \frac{\|u\|_{L^\infty(0,1)}|\lambda|^n}{\Gamma((\alpha+1)n+1)} \to 0 \text{ as } n \to \infty$$

for each $\lambda \in \mathbb{C}$ uniformly with respect to $x \in [0,1]$. Thus, passing to the limit with n in (3.22) we obtain the formula

$$u = a\sum_{k=0}^{\infty} \lambda^k I^{\alpha+k(\alpha+1)}1 - \sum_{k=0}^{\infty}\lambda^k I^{(k+1)(\alpha+1)}g. \tag{3.23}$$

We have already proven in the proof of Lemma 3.2 that the second series is uniformly convergent and

$$\sum_{k=0}^{\infty}\lambda^k I^{(k+1)(\alpha+1)}g = g * x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

The sum of the first series may be easily calculated. Indeed, from Example 2.2 we have

$$I^{\alpha+k(\alpha+1)}1 = \frac{x^{\alpha+k(\alpha+1)}}{\Gamma(1+\alpha+k(\alpha+1))},$$

hence

$$\sum_{k=0}^{\infty} \lambda^k I^{\alpha+k(\alpha+1)}1 = x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

Together, we obtain that function u given by (3.23) may be equivalently written as

$$u(x) = ax^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}) - g * x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}). \tag{3.24}$$

We may check that u given by the formula (3.24) is a solution to (3.21) and (3.20) with boundary conditions $u(0) = 0$ and $D^\alpha u(0) = a$ by a similar calculation as the one carried in the proof of Lemma 3.2. It remains to choose the value a in such a way that $u(1) = 0$. For this purpose, we take $x = 1$ in (3.24) and we obtain

$$u(1) = aE_{\alpha+1,\alpha+1}(\lambda) - (g * y^\alpha E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1).$$

To obtain that $u(1) = 0$ we choose

$$a = E_{\alpha+1,\alpha+1}(\lambda))^{-1}(g * y^\alpha E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1).$$

We note that a is well defined because, taking $\nu = \mu = \alpha + 1$ in Proposition 2.27, we obtain that $E_{\alpha+1,\alpha+1}(\lambda) \neq 0$ for λ belonging to the sector ϑ. Summing up the results we obtain that there exists a solution to (3.20) which belongs to $\bar{\mathcal{D}_\alpha}$ and it is represented by the formula

$$u(x) = \frac{(g * y^\alpha E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1)}{E_{\alpha+1,\alpha+1}(\lambda)} x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}) - g * x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

We note that here function w from the definition of $\bar{\mathcal{D}_\alpha}$ is given by

$$w(x) = \frac{(g * y^\alpha E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1)}{E_{\alpha+1,\alpha+1}(\lambda)} x^\alpha \sum_{n=1}^{\infty} \frac{\lambda^n x^{(\alpha+1)n}}{\Gamma((\alpha+1)n+\alpha+1)} - g * x^\alpha E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

In this way we proved the lemma. □

Our next aim is to prove the following.

Lemma 3.9. *For $u \in \bar{\mathcal{D}_\alpha}$ we have*

$$\operatorname{Re}(-\frac{\partial}{\partial x} D^\alpha u, u) \geq c_\alpha \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)} \tag{3.25}$$

and

$$\left|(-\frac{\partial}{\partial x} D^\alpha u, u)\right| \leq b_\alpha \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}, \tag{3.26}$$

where c_α, b_α are positive constants which depends only on α.

Proof. At first we will prove (3.25). We fix $u \in \bar{\mathcal{D}_\alpha}$. Since $u(0) = u(1) = 0$, we may integrate by parts to obtain

$$\operatorname{Re}\left(-\frac{\partial}{\partial x} D^\alpha u, u\right) = -\operatorname{Re}\int_0^1 (\frac{\partial}{\partial x} D^\alpha u)(x) \cdot \overline{u(x)} dx$$

$$= \int_0^1 D^\alpha \operatorname{Re} u(x) \cdot \frac{\partial}{\partial x} \operatorname{Re} u(x) dx + \int_0^1 D^\alpha \operatorname{Im} u(x) \cdot \frac{\partial}{\partial x} \operatorname{Im} u(x) dx.$$

We note that $\bar{\mathcal{D}_\alpha} \subseteq AC[0,1]$ hence we may apply Proposition 2.30 and we get

$$\operatorname{Re}\left(-\frac{\partial}{\partial x}D^\alpha u, u\right) = \int_0^1 D^\alpha \operatorname{Re} u(x) \cdot \partial^{1-\alpha} D^\alpha \operatorname{Re} u(x)dx + \int_0^1 D^\alpha \operatorname{Im} u(x) \cdot \partial^{1-\alpha} D^\alpha \operatorname{Im} u(x)dx.$$

By the definition of $\bar{\mathcal{D}_\alpha}$ we know that $D^\alpha u \in AC[0,1]$, hence we are allowed to apply inequality (2.19) with $w = D^\alpha \operatorname{Re} u$ and $w = D^\alpha \operatorname{Im} u$ to obtain

$$\begin{aligned}\operatorname{Re}\left(-\frac{\partial}{\partial x}D^\alpha u, u\right) &\geq c_\alpha \left\|D^\alpha u\right\|^2_{H^{\frac{1-\alpha}{2}}(0,1)} \geq c_\alpha \left\|\partial^{\frac{1-\alpha}{2}} D^\alpha u\right\|^2_{L^2(0,1)} \\ &= c_\alpha \left\|D^{\frac{1+\alpha}{2}} u\right\|^2_{L^2(0,1)} = c_\alpha \left\|\partial^{\frac{1+\alpha}{2}} u\right\|^2_{L^2(0,1)} \geq c_\alpha \left\|u\right\|_{H^{\frac{1+\alpha}{2}}(0,1)}.\end{aligned}$$

Here in the second inequality we used Proposition 2.32, the first equality follows from Proposition 2.30, the second equality is a consequence of the fact that u vanishes at zero and the last inequality comes from Proposition 2.32. It remains to show (3.26). We make use of Remark 2.7, Definition 2.18 and Proposition 2.30 to arrive at the following sequence of identities

$$\frac{\partial}{\partial x}D^\alpha u = \partial^{\frac{1+\alpha}{2}} \partial^{\frac{1-\alpha}{2}} D^\alpha u = \frac{\partial}{\partial x} I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u.$$

Then, we apply integration by parts formula and identity (2.23) to get

$$\int_0^1 \frac{\partial}{\partial x}D^\alpha u \cdot \bar{u}dx = -\int_0^1 I^{\frac{1-\alpha}{2}} D^{\frac{1+\alpha}{2}} u \cdot \bar{u}_x dx = -\int_0^1 D^{\frac{1+\alpha}{2}} u \cdot I_-^{\frac{1-\alpha}{2}} \bar{u}_x dx.$$

We note that the boundary terms in integration by parts formula vanish due to $u(0) = u(1) = 0$. Finally, we get

$$\begin{aligned}\left|\int_0^1 \frac{\partial}{\partial x}D^\alpha u \cdot \bar{u}dx\right| &\leq \left\|D^{\frac{1+\alpha}{2}} u\right\|_{L^2(0,1)} \left\|D_-^{\frac{1+\alpha}{2}} \bar{u}\right\|_{L^2(0,1)} \\ &= \left\|\partial^{\frac{1+\alpha}{2}} u\right\|_{L^2(0,1)} \left\|\partial_-^{\frac{1+\alpha}{2}} \bar{u}\right\|_{L^2(0,1)} \leq b_\alpha \left\|u\right\|^2_{H^{\frac{1+\alpha}{2}}}.\end{aligned}$$

We note that here we again used the fact that u vanishes at the boundary and we applied Proposition 2.32. $\square$

Now we are able to state the result.

Theorem 3.10. *The operator $\frac{\partial}{\partial x}D^\alpha : \bar{\mathcal{D}_\alpha} \subseteq L^2(0,1) \to L^2(0,1)$ is a densely defined sectorial operator, thus it generates an analytic semigroup.*

Proof. Once we established Lemma 3.8 and Lemma 3.9, the argument is identical as in the proof of Theorem 3.5. $\square$

Remark 3.3. *Theorem 3.10 allows us to apply Theorem 2.11 to obtain existence and regularity results for a solution to (3.18) with $f \equiv 0$. We note, that in this case, the solution has a singularity at the left endpoint of the interval of the form x^α. Hence, in general the solution to this problem is less regular then in the case of boundary condition $u_x(0,t) = 0$.*

3.2.1. Non-homogenous boundary conditions

We finish this section with a remark about the case with non-homogenous boundary conditions. Let us discuss

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^\alpha u = f & \text{in } (0,1)\times(0,T), \\ u(0,t)=g(t), \quad u(1,t)=h(t) & \text{for } t\in(0,T), \\ u(x,0)=u_0(x) & \text{in } (0,1), \end{cases} \tag{3.27}$$

where $g,h \in C^{1,\nu}[0,T]$ and $f \in C^{0,\nu}([0,T];L^2(0,1))$ for a $0<\nu<1$. Then we may define

$$v(x,t) := u(x,t) - g(t)\eta(x) - h(t)\varphi(x), \tag{3.28}$$

where η and φ are smooth functions such that $\eta\equiv 1$, $\varphi\equiv 0$ near the left endpoint of the interval and $\eta\equiv 0$, $\varphi\equiv 1$ near the right endpoint of the interval. We rewrite the system (3.27) in terms of function v

$$\begin{cases} v_t - \frac{\partial}{\partial x}D^\alpha v = \bar{f} & \text{in } (0,1)\times(0,T), \\ v(0,t)=0, \quad v(1,t)=0 & \text{for } t\in(0,T), \\ v(x,0)=v_0(x) & \text{in } (0,1), \end{cases} \tag{3.29}$$

where

$$\bar{f} = f - g'(t)\eta(x) - h'(t)\varphi(x) + g(t)\frac{\partial}{\partial x}D^\alpha\eta(x) + h(t)\frac{\partial}{\partial x}D^\alpha\varphi(x)$$

and

$$v_0(x) = u_0(x) - g(0)\eta(x) - h(0)\varphi(x).$$

We note that $\varphi_x, \eta_x \in {}_0H^\alpha(0,1)$, hence $\frac{\partial}{\partial x}D^\alpha\varphi \in L^2(0,1)$ and $\frac{\partial}{\partial x}D^\alpha\varphi \in L^2(0,1)$. Thus, we may apply the standard theory of analytic semigroups to obtain existence and regularity results for this problem. For instance, since $\bar{f} \in C^{0,\nu}([0,T];L^2(0,1))$ we may apply Theorem 2.12 to obtain basic result concerning the existence of the solution to (3.29). Then, we recover the solution to (3.27) from identity (3.28).

3.3. Case with prescribed flux on the left boundary

In this section we will consider the problem with prescribed nonlocal flux on the left boundary, i.e.

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^\alpha u = f & \text{in } (0,1)\times(0,T), \\ (D^\alpha u)(0,t)=h(t), \quad u(1,t)=0 & \text{for } t\in(0,T), \\ u(x,0)=u_0(x) & \text{in } (0,1). \end{cases} \tag{3.30}$$

We note that the condition on $(D^\alpha u)(0)$ is connected with regularity of function u. Indeed, we have the following.

Lemma 3.11. *Let F be an absolutely continuous function and $f := F'$. Then we denote*

$$(D^\alpha F)(0) := \lim_{x\to 0}\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha}f(p)dp.$$

1. *If* $(D^\alpha F)(0)$ *exists and* $(D^\alpha F)(0) = c$, *then* $\lim_{y\to 0}\frac{F(y)}{y^\alpha} = \frac{c}{\Gamma(1+\alpha)}$,
2. *if the limit* $\lim_{y\to 0}\frac{f(y)}{y^{\alpha-1}}$ *exists and* $\lim_{y\to 0}\frac{f(y)}{y^{\alpha-1}} = \frac{c}{\Gamma(\alpha)}$, *then* $(D^\alpha F)(0) = c$.

Proof. Let us firstly assume that

$$\lim_{x\to 0}\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha} f(p)dp = c.$$

We fix $\varepsilon > 0$. Then, there exists $x_0 > 0$ such that for every $0 \le y < x_0$ there holds

$$c - \varepsilon \le \frac{1}{\Gamma(1-\alpha)}\int_0^y (y-p)^{-\alpha} f(p)dp \le \varepsilon + c.$$

We apply I^α to these inequalities. By Example 2.2 and Proposition 2.22 we get

$$(c-\varepsilon)\frac{y^\alpha}{\Gamma(1+\alpha)} \le F(y) \le (c+\varepsilon)\frac{y^\alpha}{\Gamma(1+\alpha)} \text{ for every } y < x_0,$$

which is equivalent with

$$\left|\Gamma(1+\alpha)\frac{F(y)}{y^\alpha} - c\right| \le \varepsilon \text{ for every } y < x_0.$$

Hence, $\lim_{y\to 0}\frac{F(y)}{y^\alpha} = \frac{c}{\Gamma(1+\alpha)}$. Now, we assume that $\lim_{y\to 0}\frac{f(y)}{y^{\alpha-1}} = \frac{c}{\Gamma(\alpha)}$. Then, we obtain that for any fixed $\varepsilon > 0$, there exists $x_0 > 0$ such that for all $0 < y < x_0$

$$\left(\frac{c}{\Gamma(\alpha)} - \varepsilon\right)y^{\alpha-1} \le f(y) \le \left(\frac{c}{\Gamma(\alpha)} + \varepsilon\right)y^{\alpha-1}.$$

Applying $I^{1-\alpha}$ to the inequalities above we obtain for every $0 < y < x_0$ that

$$c - \varepsilon\Gamma(\alpha) \le (I^{1-\alpha} f)(y) \le c + \varepsilon\Gamma(\alpha),$$

where we made use of Example 2.1. Since $\varepsilon > 0$ is arbitrary we obtain the claim. □

In view of Lemma 3.11 it is natural to search for a solution to (3.30) in the form

$$u = \frac{h(t)}{\Gamma(1+\alpha)}x^\alpha + v, \quad v_x \in {}_0H^\alpha(0,1). \tag{3.31}$$

Then, we may rewrite problem (3.30) in terms of function v. Namely,

$$\begin{cases} v_t - \frac{\partial}{\partial x}D^\alpha v = f - \frac{h'(t)}{\Gamma(1+\alpha)}x^\alpha & \text{in } (0,1)\times(0,T), \\ v_x \in {}_0H^\alpha(0,1), \quad v(1,t) = -\frac{h(t)}{\Gamma(1+\alpha)} & \text{for } t \in (0,T), \\ v(x,0) = u_0(x) - \frac{h(0)}{\Gamma(1+\alpha)}x^\alpha & \text{in } (0,1). \end{cases} \tag{3.32}$$

Then, we introduce

$$w(x,t) = v(x,t) + \frac{h(t)}{\Gamma(1+\alpha)}\varphi(x), \tag{3.33}$$

where φ is a smooth function such that $\varphi \equiv 0$ near the left endpoint of the interval and $\varphi \equiv 1$ near the right endpoint of the interval. We may rewrite system (3.32) in terms of function w, i.e.

$$\begin{cases} w_t - \frac{\partial}{\partial x}D^\alpha w = \bar{f} & \text{in } (0,1)\times(0,T), \\ w_x \in {}_0H^\alpha(0,1), \quad w(1,t) = 0 & \text{for } t \in (0,T), \\ w(x,0) = w_0(x) & \text{in } (0,1), \end{cases} \tag{3.34}$$

where

$$\bar{f} = f - \frac{h'(t)}{\Gamma(1+\alpha)}(x^\alpha - \varphi(x)) - \frac{h(t)}{\Gamma(1+\alpha)}\frac{\partial}{\partial x}D^\alpha\varphi,$$

$$w_0(x) = u_0(x) + \frac{h(0)}{\Gamma(1+\alpha)}(\varphi(x) - x^\alpha).$$

In order to solve (3.34) we may apply Theorem 3.5 and the standard theory of analytic semigroups. For example, assuming $f \in C^{0,\nu}(0,T;L^2(0,1))$, $h \in C^{1,\nu}([0,T])$, $u_0 \in L^2(0,1)$, we may apply Theorem 2.12. Then, we can recover a solution to (3.30) from the identities (3.31) and (3.33). We summarize the obtained result in the following theorem.

Theorem 3.12. *Let us assume that $f \in C^{0,\nu}(0,T;L^2(0,1))$, $h \in C^{1,\nu}([0,T])$ and $u_0 \in L^2(0,1)$. Then there exists a solution to (3.30) such that for every $t \in (0,T]$ the equation $(3.30)_1$ is satisfied in $L^2(0,1)$ and there hold $u(1,t) = 0$, $(D^\alpha u)(0,t) = h(t)$. Furthermore, $u \in C([0,T];L^2(0,1)) \cap C^1((0,T];L^2(0,1))$ and for every $\varepsilon > 0$ $u \in C((0,T];H^{1+\alpha}(\varepsilon,1))$.*

3.4. Case with less regular source term

3.4.1. Motivation

Studying the results from previous sections, we may infer, that one may pass from the problem with non-homogenous Dirichlet boundary conditions to the homogenous problem rather painless. Indeed, it is enough to use an appropriate auxiliary function and transform the problem to the one with the source term that is square integrable with respect to space variable. However, we note that the situation appears to be different if we deal with the problem with non-homogenous Neumann condition

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^\alpha u = 0 & \text{in } (0,1)\times(0,T), \\ u_x(0,t) = h(t), \quad u(1,t) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1). \end{cases} \tag{3.35}$$

We would like to somehow transform this problem to (3.3). To this end, we fix a smooth function ρ such that $\rho'(0) = 1$, $\rho(1) = 0$ and we introduce an auxiliary function v as follows

$$v(x,t) = u(x,t) - h(t)\rho(x). \tag{3.36}$$

Then, applying identity (3.1) and Remark 2.6 we obtain

$$\frac{\partial}{\partial x}D^\alpha \rho = \partial^\alpha \rho_x = D^\alpha \rho_x + \frac{x^{-\alpha}}{\Gamma(1-\alpha)}.$$

Hence, our problem may be reduced to

$$\begin{cases} v_t - \frac{\partial}{\partial x}D^\alpha v = -h'(t)\rho + h(t)D^\alpha\rho_x + \frac{h(t)}{\Gamma(1-\alpha)}x^{-\alpha} & \text{in } (0,1)\times(0,T), \\ v_x(0,t) = 0, \quad v(1,t) = 0 & \text{for } t \in (0,T), \\ v(x,0) = u_0(x) - h(0)\rho(x) =: v_0(x) & \text{in } (0,1). \end{cases} \tag{3.37}$$

We note that the last component of the source function has a singularity and in the case of $\alpha \in [\frac{1}{2}, 1)$ the singularity is not square integrable. Hence, we are not allowed to use already obtained results. This motivates us to investigate the problem (3.3) with f that does not belong to $L^2(0,1)$. Although the analytic semigroup theory is a powerful tool in the theory of existence and regularity of solutions to parabolic type problems, it is not very flexible. That is way, here we would like to present a different approach, based on energy estimates. We will find weak solutions to the problem (3.3) with rough regularity of the source term, thus in the case when we are not able to apply directly the semigroup theory.

3.4.2. Energy method

Our goal is to solve the following problem

$$\begin{cases} v_t - \frac{\partial}{\partial x} D^\alpha v = f & \text{in } (0,1) \times (0,T), \\ v_x \in {}_0H^\alpha(0,1), \quad v(1,t) = 0 & \text{for } t \in (0,T), \\ v(x,0) = v_0(x) & \text{in } (0,1), \end{cases} \tag{3.38}$$

where $f \in L^2(0,T; L^1(0,1))$. We will solve this problem by approximation. Let us take a sequence $f^\varepsilon \in C^1([0,T]; \mathcal{D}_\alpha)$ such that $f^\varepsilon \to f$ in $L^2(0,T; L^1(0,1))$. Let us show that such sequence exists. There exists a sequence of simple functions $s_k = \sum_{n=0}^k \chi_{E_n}(t) g_n$, where $g_n \in L^1(0,1)$ and E_n are measurable subsets of $(0,T)$, such that $s_k \to f$ in $L^2(0,T; L^1(0,1))$. We define a sequence $f^{k,m,\delta}$ as follows

$$f^{k,m,\delta}(x,t) = \sum_{n=0}^k \eta_\delta * \chi_{E_n}(t) g_n^m,$$

where η_δ denotes a standard mollifier and $\{g_n^m\} \subseteq C_0^\infty(0,1)$ is such that for every n we have $g_n^m \to g_n$ as $m \to \infty$ in $L^1(0,1)$. Then $f^{k,m,\delta} \in C^\infty([0,T]; C_0^\infty(0,1))$ and $f^{k,m,\delta} \to f$ in $L^2(0,T; L^1(0,1))$ as $m \to \infty$, $k \to \infty$ and $\delta \to 0$. Let us assume that $v_0 \in L^2(0,1)$. By Theorem 2.12 and Theorem 3.5 we obtain that there exists exactly one solution to approximate problem

$$\begin{cases} v_t^\varepsilon - \frac{\partial}{\partial x} D^\alpha v^\varepsilon = f^\varepsilon & \text{in } (0,1) \times (0,T), \\ v_x^\varepsilon \in {}_0H^\alpha(0,1), \quad v^\varepsilon(1,t) = 0 & \text{for } t \in (0,T), \\ v^\varepsilon(x,0) = v_0(x) & \text{in } (0,1), \end{cases} \tag{3.39}$$

which belongs to $C([0,T]; L^2(0,1)) \cap C((0,T]; \mathcal{D}_\alpha) \cap C^1((0,T]; L^2(0,1))$. Let us denote by $T(t)$ the analytic semigroup generated by $\frac{\partial}{\partial x} D^\alpha$ given by Theorem 3.5. Then, the solution to (3.39) is given by the formula

$$v^\varepsilon(x,t) = T(t) v_0(x) + \int_0^t T(t-\tau) f^\varepsilon(x,\tau) d\tau.$$

We note that the interpolation space between L^2 and $\mathcal{D}_\alpha$ is characterized as follows

$[L^2(0,1), \mathcal{D}_\alpha]_\theta =$

$$= \begin{cases} {}^0H^{(1+\alpha)\theta}(0,1) & \text{if} \quad \theta \in (0, \min\{1, \frac{3}{2(1+\alpha)}\}), \\ \{u \in H^{(1+\alpha)\theta}(0,1) : u(1) = 0, \;\; u_x \in {}_0H^{(1+\alpha)\theta-1}(0,1)\} & \text{if} \quad \theta \in [\min\{1, \frac{3}{2(1+\alpha)}\}, 1]. \end{cases} \tag{3.40}$$

For clarity, let us describe how the interpolation space looks like in dependence of α and θ.

$[L^2(0,1), \mathcal{D}_\alpha]_\theta =$

$$= \begin{cases} H^{(1+\alpha)\theta}(0,1) & \text{if} \quad \theta \in (0, \frac{1}{2(1+\alpha)}), \;\; \alpha \in (0,1), \\ {}^0H^{\frac{1}{2}}(0,1) & \text{if} \quad \theta = \frac{1}{2(1+\alpha)}, \;\; \alpha \in (0,1), \\ \{u \in H^{(1+\alpha)\theta}(0,1) : u(1) = 0\} & \text{if} \quad \theta \in (\frac{1}{2(1+\alpha)}, 1), \;\; \alpha \in (0, \frac{1}{2}], \\ \{u \in H^{(1+\alpha)\theta}(0,1) : u(1) = 0\} & \text{if} \quad \theta \in (\frac{1}{2(1+\alpha)}, \frac{3}{2(1+\alpha)}), \;\; \alpha \in (\frac{1}{2}, 1), \\ \{u \in H^{\frac{3}{2}}(0,1) : u(1) = 0, \;\; u_x \in {}_0H^{\frac{1}{2}}(0,1)\} & \text{if} \quad \theta = \frac{3}{2(1+\alpha)}, \;\; \alpha \in (\frac{1}{2}, 1), \\ \{u \in H^{(1+\alpha)\theta}(0,1) : u(1) = 0, \;\; u_x(0) = 0\} & \text{if} \quad \theta \in (\frac{3}{2(1+\alpha)}, 1), \;\; \alpha \in (\frac{1}{2}, 1). \end{cases}$$

Furthermore, for every $\gamma \in (0, 1+\alpha)$, $\alpha \in (0,1)$ and $g \in [L^2(0,1), \mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}$ we have $\|g\|_{H^\gamma(0,1)} \le \|g\|_{[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}}$ and in the case $\gamma \notin \{\frac{1}{2}, \frac{3}{2}\}$ the inequality is in fact the equality. We note that $\frac{\partial}{\partial x}D^\alpha$ is sectorial, hence in particular it is closed. Thus, [6, Proposition C.4] allows us to pass with $\frac{\partial}{\partial x}D^\alpha$ under the integral sign. Hence, by Proposition 2.13, for every $0 < \gamma < 1+\alpha$ we have

$$\left\| \frac{\partial}{\partial x} D^\alpha v^\varepsilon(\cdot,t) \right\|_{H^\gamma(0,1)} \le \left\| \frac{\partial}{\partial x} D^\alpha v^\varepsilon(\cdot,t) \right\|_{[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}}$$

$$\le \left\| \frac{\partial}{\partial x} D^\alpha T(t) v_0 \right\|_{[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} + \int_0^t \left\| \frac{\partial}{\partial x} D^\alpha T(t-\tau) f^\varepsilon(\cdot,\tau) \right\|_{[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} d\tau$$

$$\le ct^{-\frac{\gamma}{1+\alpha}-1} \|v_0\|_{L^2(0,1)} + c\int_0^t (t-\tau)^{-\frac{\gamma}{1+\alpha}} \|f^\varepsilon(\cdot,\tau)\|_{\mathcal{D}_\alpha} d\tau$$

$$\le ct^{-\frac{\gamma}{1+\alpha}-1} \|v_0\|_{L^2(0,1)} + ct^{1-\frac{\gamma}{1+\alpha}} \|f^\varepsilon\|_{C([0,T];\mathcal{D}_\alpha)}.$$

In view of (3.1), we obtained that $\partial^\alpha v_x^\varepsilon \in L^\infty_{loc}(0,T; H^\gamma(0,1))$ for every $0 < \gamma < 1+\alpha$. Since $\partial^\alpha v_x^\varepsilon \in C((0,T]; L^2(0,1))$, applying the interpolation estimate ([19, Corollary 1.2.7]) we obtain that for every $0 < t, \tau \le T$, $0 < \gamma < \gamma_1 < 1+\alpha$

$$\|\partial^\alpha v_x^\varepsilon(\cdot,t) - \partial^\alpha v_x^\varepsilon(\cdot,\tau)\|_{H^\gamma(0,1)}$$

$$\le c(\gamma,\gamma_1) \|\partial^\alpha v_x^\varepsilon(\cdot,t) - \partial^\alpha v_x^\varepsilon(\cdot,\tau)\|_{L^2(0,1)}^{1-\frac{\gamma}{\gamma_1}} \|\partial^\alpha v_x^\varepsilon(\cdot,t) - \partial^\alpha v_x^\varepsilon(\cdot,\tau)\|_{H^{\gamma_1}(0,1)}^{\frac{\gamma}{\gamma_1}}.$$

The last norm is bounded on every compact interval contained in $(0,T]$ while the first norm on the r.h.s. tends to zero as $t \to \tau$. Thus,

$$\partial^\alpha v_x^\varepsilon \in C((0,T]; H^\gamma(0,1)) \quad \text{for every} \quad 0 < \gamma < 1+\alpha. \tag{3.41}$$

Furthermore, from Corollary 2.33 and the identity

$$v_x^\varepsilon = I^\alpha(\partial^\alpha v_x^\varepsilon - \partial^\alpha v_x^\varepsilon(0)) + \partial^\alpha v_x^\varepsilon(0)\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{3.42}$$

we infer that for every $\varepsilon_1 \in (0,1)$ there holds

$$v_x^\varepsilon \in C((0,T]; H^{\gamma+\alpha}(\varepsilon_1, 1)) \quad \text{for every} \quad 0<\gamma<1+\alpha. \tag{3.43}$$

Our aim is to pass to the limit with ε and obtain a weak solution to (3.38). At first we will prove the following result.

Theorem 3.13. *Let us consider the problem (3.38) with $v_0 \in L^2(0,1)$, $f \in L^2(0,T;L^1(0,1))$. Then, there exists*

$$v \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;{}^0H^{\frac{1+\alpha}{2}}(0,1)), \;\; v_t \in L^2(0,T;({}^0H^{\frac{1+\alpha}{2}}(0,1))')$$

such that for every $w \in {}^0H^{\frac{1+\alpha}{2}}(0,1)$ and every $\Psi \in C_0^\infty(0,T)$ there holds

$$\int_0^T \langle v_t, w\rangle_{({}^0H^{\frac{1+\alpha}{2}}(0,1))' \times {}^0H^{\frac{1+\alpha}{2}}(0,1)} \Psi dt$$

$$= \int_0^T \int_0^1 I^{\frac{1-\alpha}{2}} v_x \cdot \partial_-^{\frac{1+\alpha}{2}} w dx \Psi dt + \int_0^T \int_0^1 f \cdot w dx \Psi dt, \tag{3.44}$$

where $I^{\frac{1-\alpha}{2}}$ is understood in the sense of extension given by Proposition 2.35.

Proof. We multiply (3.39) by v^ε and integrate with respect to space. Then, we arrive at

$$\int_0^1 v_t^\varepsilon \cdot v^\varepsilon dx - \int_0^1 \frac{\partial}{\partial x} D^\alpha v^\varepsilon \cdot v^\varepsilon dx = \int_0^1 f^\varepsilon \cdot v^\varepsilon dx \quad \text{for every } t \in (0,T).$$

Since $v^\varepsilon \in C((0,T];\mathcal{D}_\alpha)$ we may apply inequality (3.12) and Lemma 3.3 to obtain that

$$-\int_0^1 \frac{\partial}{\partial x} D^\alpha v^\varepsilon \cdot v^\varepsilon dx \geq c_\alpha \left\| v^\varepsilon \right\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}.$$

Hence, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |v^\varepsilon|^2 dx + c_\alpha \left\| v^\varepsilon \right\|^2_{H^{\frac{1+\alpha}{2}}(0,1)} \leq \left\| f^\varepsilon \right\|_{L^1(0,1)} \left\| v^\varepsilon \right\|_{L^\infty(0,1)}.$$

We apply the Sobolev and Young inequalities and then, we integrate with respect to time to get

$$\frac{1}{2}\int_0^1 |v^\varepsilon(x,t)|^2 dx + \frac{c_\alpha}{2}\int_0^t \left\| v^\varepsilon(\cdot,\tau) \right\|^2_{H^{\frac{1+\alpha}{2}}(0,1)} d\tau \leq \frac{1}{2}\left\| v_0 \right\|^2_{L^2(0,1)} + c(\alpha) \left\| f^\varepsilon \right\|^2_{L^2(0,T;L^1(0,1))}. \tag{3.45}$$

Since $f^\varepsilon \to f$ in $L^2(0,T;L^1(0,1))$, the sequence $\{v^\varepsilon\}$ is bounded in $L^\infty(0,T;L^2(0,1))$ and in $L^2(0,T;H^{\frac{1+\alpha}{2}}(0,1))$. We will show the estimates for v_t^ε. Let $w \in {}^0H^{\frac{1+\alpha}{2}}(0,1)$ and we choose a sequence $w_k \in {}^0C^\infty(0,1)$, such that $w_k \to w$ in ${}^0H^{\frac{1+\alpha}{2}}(0,1)$. We multiply (3.39) by w_k and integrate with respect to space

$$\int_0^1 v_t^\varepsilon \cdot w_k dx = \int_0^1 \frac{\partial}{\partial x} D^\alpha v^\varepsilon \cdot w_k dx + \int_0^1 f^\varepsilon \cdot w_k dx.$$

We recall that $v^\varepsilon \in C([0,T];\mathcal{D}_\alpha)$. Since w_k are smooth and $w_k(1)=0$, we may apply the inequality (3.14) to obtain

$$\left| \int_0^1 v_t^\varepsilon \cdot w_k dx \right| \leq b_\alpha \left\| v^\varepsilon \right\|_{H^{\frac{1+\alpha}{2}}(0,1)} \left\| w_k \right\|_{{}^0H^{\frac{1+\alpha}{2}}(0,1)} + \left\| f^\varepsilon \right\|_{L^1(0,1)} \left\| w_k \right\|_{L^\infty(0,1)}.$$

Since $v_t^\varepsilon \in C((0,T];L^2(0,1))$ we may pass to the limit with k to get

$$\left| \int_0^1 v_t^\varepsilon \cdot w dx \right| \leq b_\alpha \left\| v^\varepsilon \right\|_{H^{\frac{1+\alpha}{2}}(0,1)} \left\| w \right\|_{{}^0H^{\frac{1+\alpha}{2}}(0,1)} + \left\| f^\varepsilon \right\|_{L^1(0,1)} \left\| w \right\|_{L^\infty(0,1)}.$$

Taking the supremum over w such that $\|w\|_{{}^0H^{\frac{1+\alpha}{2}}(0,1)} = 1$ we get

$$\|v_t^\varepsilon\|_{({}^0H^{\frac{1+\alpha}{2}}(0,1))'} \le b_\alpha \|v^\varepsilon\|_{H^{\frac{1+\alpha}{2}}(0,1)} + c(\alpha)\|f\|_{L^1(0,1)}.$$

We raise both sides to power two and integrate with respect to time. In consequence, we obtain that $\{v_t^\varepsilon\}$ is bounded in $L^2(0,T;({}^0H^{\frac{1+\alpha}{2}}(0,1))')$. Now we will pass to the limit. We fix $w \in {}^0H^{\frac{1+\alpha}{2}}(0,1)$. Then, there exists a sequence $\{w_k\} \subseteq {}^0C^\infty(0,1)$ such that $w_k \to w$ in ${}^0H^{\frac{1+\alpha}{2}}(0,1)$. We multiply (3.39) by w_k and we integrate it with respect to space. Then, we multiply the identity by $\Psi \in C_0^\infty(0,T)$ and integrate with respect to time

$$\int_0^T\int_0^1 v_t^\varepsilon \cdot w_k dx\Psi dt = \int_0^T\int_0^1 \frac{\partial}{\partial x}D^\alpha v^\varepsilon \cdot w_k dx\Psi dt + \int_0^T\int_0^1 f^\varepsilon \cdot w_k dx\Psi dt.$$

Since w_k are smooth and $w_k(1) = 0$, we may apply identity (3.15) to obtain

$$\int_0^1 \frac{\partial}{\partial x}D^\alpha v^\varepsilon \cdot w_k dx = \int_0^1 D^{\frac{1+\alpha}{2}}v^\varepsilon \cdot \partial_-^{\frac{1+\alpha}{2}} w_k dx.$$

By Proposition 2.32 the operator $\partial_-^{\frac{1+\alpha}{2}}$ is linear and bounded from ${}^0H^{\frac{1+\alpha}{2}}(0,1)$ to $L^2(0,1)$. Hence, passing to the limit with k we obtain

$$\int_0^T\int_0^1 v_t^\varepsilon \cdot w dx\Psi dt = \int_0^T\int_0^1 D^{\frac{1+\alpha}{2}}v^\varepsilon \cdot \partial_-^{\frac{1+\alpha}{2}} w dx\Psi dt + \int_0^T\int_0^1 f^\varepsilon \cdot w dx\Psi dt.$$

We will proceed with each term separately. Since $f^\varepsilon \to f$ in $L^2(0,T;L^1(0,1))$ we get

$$\int_0^T\int_0^1 f^\varepsilon \cdot w dx\Psi dt \to \int_0^T\int_0^1 f \cdot w dx\Psi dt.$$

Applying the weak-compactness argument we obtain that there exists $\chi \in L^2(0,T;({}^0H^{\frac{1+\alpha}{2}}(0,1))')$ such that on the subsequence

$$v_t^\varepsilon \rightharpoonup \chi \quad \in L^2(0,T;({}^0H^{\frac{1+\alpha}{2}}(0,1))').$$

Hence,

$$\int_0^T\int_0^1 v_t^\varepsilon \cdot w dx\Psi dt \to \int_0^T \langle \chi, w\rangle_{({}^0H^{\frac{1+\alpha}{2}}(0,1))' \times {}^0H^{\frac{1+\alpha}{2}}(0,1)} \Psi dt.$$

On the other hand, we have

$$\int_0^T\int_0^1 v_t^\varepsilon \cdot w dx\Psi dt = -\int_0^T\int_0^1 v^\varepsilon \cdot w dx\Psi' dt \to \int_0^T\int_0^1 v \cdot w dx\Psi' dt.$$

Hence, χ is a weak derivative of v and we have

$$\int_0^T\int_0^1 v_t^\varepsilon \cdot w dx\Psi dt \to \int_0^T\int_0^1 \langle v_t, w\rangle_{({}^0H^{\frac{1+\alpha}{2}}(0,1))' \times {}^0H^{\frac{1+\alpha}{2}}(0,1)} \Psi dt,$$

where the time derivative is understood in a weak sense. Now we will pass to the limit in the last term. Applying estimate (3.45) together with Lemma 3.3 we obtain that the sequence $D^{\frac{1+\alpha}{2}}v^\varepsilon$ is bounded in $L^2(0,T;L^2(0,1))$ and hence, on a subsequence

$$\int_0^T\int_0^1 D^{\frac{1+\alpha}{2}}v^\varepsilon \cdot \partial_-^{\frac{1+\alpha}{2}} w dx\Psi dt \to \int_0^T\int_0^1 \Phi \cdot \partial_-^{\frac{1+\alpha}{2}} w dx\Psi dt,$$

where $\Phi \in L^2(0,T;L^2(0,1))$. We are going to characterize this limit. Since $D^{\frac{1+\alpha}{2}}v^\varepsilon = I^{\frac{1-\alpha}{2}}v_x^\varepsilon$ and $v_x^\varepsilon \rightharpoonup v_x$ in $L^2(0,T;H^{\frac{\alpha-1}{2}}(0,1))$ we can write

$$\int_0^T\int_0^1 D^{\frac{1+\alpha}{2}}v^\varepsilon \cdot \partial_-^{\frac{1+\alpha}{2}} w dx\Psi dt = \int_0^T\int_0^1 v_x^\varepsilon \cdot I_-^{\frac{1-\alpha}{2}}\partial_-^{\frac{1+\alpha}{2}} w dx\Psi dt$$

$$\to \int_0^T \left\langle v_x, I_-^{\frac{1-\alpha}{2}}\partial_-^{\frac{1+\alpha}{2}} w\right\rangle_{H^{\frac{\alpha-1}{2}}(0,1)\times H^{\frac{1-\alpha}{2}}(0,1)} \Psi dt$$

$$= \int_0^T \int_0^1 I^{\frac{1-\alpha}{2}} v_x \cdot \partial_-^{\frac{1+\alpha}{2}} w dx \Psi dt,$$

where $I^{\frac{1-\alpha}{2}}$ denotes an extension of fractional integral on a dual space given by Proposition 2.35. This way we obtained (3.44) and the proof is finished. □

Motivated by the problem (3.35) we would like to investigate how we can increase the regularity of the solutions to (3.38) if the source term has better regularity away from left endpoint of the interval. We formulate the result in the next theorem.

Theorem 3.14. *If $v_0 \in H^1(0,1)$, $f \in L^2(0,T;L^1(0,1)) \cap L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))$ for fixed $\varepsilon_1 \in (0,1)$, then weak solution to (3.38), obtained in Theorem 3.13, satisfies additionally for every $\delta \in (\varepsilon_1, 1)$*

$$v \in L^\infty((0,T);H^1(\delta,1)), \;\; v_x \in L^2((0,T);H^{\frac{1+\alpha}{2}}(\delta,1)), \;\; v_t \in L^2((0,T);H^{\frac{1-\alpha}{2}}(\delta,1)).$$

Moreover, for every $\varepsilon_1 < \delta_1 < \delta < 1$, $w \in L^2(\delta,1)$ and every $\Psi \in C_0^\infty(0,T)$, there holds

$$\int_0^T \int_\delta^1 v_t \cdot w dx \Psi dt = \int_0^T \int_\delta^1 f \cdot w dx \Psi dt$$

$$+ \int_0^T \int_\delta^1 [\frac{\partial}{\partial x} I_{\delta_1}^{1-\alpha} v_x + \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \left\langle (x-\cdot)^{-\alpha}, v_x \right\rangle_{H^{\frac{1-\alpha}{2}}(0,\delta_1) \times H^{\frac{\alpha-1}{2}}(0,\delta_1)}] w dx \Psi dt,$$

where, $I_a^\beta f(x) := \frac{1}{\Gamma(\beta)} \int_a^x (x-p)^{\beta-1} f(p) dp$.

Proof. We choose a sequence $f^\varepsilon \in C^1([0,T];\mathcal{D}_\alpha)$, such that $f^\varepsilon \to f$ in $L^2(0,T;L^1(0,1))$ and $f^\varepsilon \to f$ in $L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))$. Let us briefly justify that such sequence exists. As it was shown at the beginning of section 3.2.4 we may choose a sequence $\{f_1^\varepsilon\} \subseteq C^1([0,T;C_0^\infty(0,\varepsilon_1)])$ such that $f_1^\varepsilon \to f$ in $L^2(0,T;L^1(0,\varepsilon_1))$ and a sequence $\{f_2^\varepsilon\} \subseteq C^1([0,T;C_0^\infty(\varepsilon_1,1)])$ such that $f_2^\varepsilon \to f$ in $L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))$, where we used the fact that $\frac{1-\alpha}{2} < \frac{1}{2}$. Then the sequence $\{f^\varepsilon\} \equiv f_1^\varepsilon$ on $[0,T] \times [0,\varepsilon_1]$ and $f^\varepsilon \equiv f_2^\varepsilon$ on $[0,T] \times [\varepsilon_1, 1]$ fulfills the assumptions.

We denote by v^ε the solution to (3.39) given by analytic semigroup generated by $\frac{\partial}{\partial x} D^\alpha$. We note that a sequence v^ε satisfies the estimate (3.45) and on the subsequence it converges weakly* in $L^\infty(0,T;L^2(0,1))$ and weakly in $L^2(0,T;H^{\frac{1+\alpha}{2}}(0,1))$ to a weak solution v to (3.38) obtained in Theorem 3.13. We fix $\varepsilon_1 < \delta < 1$. Let $\eta \geq 0$ be an arbitrary smooth function such that $\eta \equiv 0$ on $[0, (\varepsilon_1 + \delta)/2]$, $\eta \equiv 1$ on $[\delta, 1]$. At first we will show that

$$\eta \cdot \frac{\partial}{\partial x} D^\alpha v^\varepsilon = \frac{\partial}{\partial x} D^\alpha (v^\varepsilon \cdot \eta) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha-1} (\eta(x) - \eta(p)) v_x^\varepsilon(p) dp - \partial^\alpha (\eta' \cdot v^\varepsilon). \tag{3.46}$$

Indeed, applying identity (3.1) and Proposition 2.26 we arrive at

$$\eta \cdot \frac{\partial}{\partial x} D^\alpha v^\varepsilon = \eta \cdot \partial^\alpha v_x^\varepsilon = \partial^\alpha (v_x^\varepsilon \cdot \eta) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha-1} (\eta(x) - \eta(p)) v_x^\varepsilon(p) dp$$

$$= \partial^\alpha (v^\varepsilon \cdot \eta)_x - \partial^\alpha (v^\varepsilon \cdot \eta') - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha-1} (\eta(x) - \eta(p)) v_x^\varepsilon(p) dp.$$

Applying again identity (3.1) we arrive at (3.46). We multiply (3.39) by $(v^{\varepsilon}\eta)_{xx}\cdot\eta$ and integrate it over $(0,1)$. We note that by (3.43) we have $(v^{\varepsilon}\eta)_{xx}(\cdot,t)\in L^2(0,1)$ for all $t\in(0,T)$. We obtain

$$\int_0^1 v_t^{\varepsilon}\eta\cdot(v^{\varepsilon}\eta)_{xx}dx-\int_0^1\frac{\partial}{\partial x}D^{\alpha}v^{\varepsilon}\cdot\eta\cdot(v^{\varepsilon}\eta)_{xx}dx=\int_0^1 f^{\varepsilon}\eta\cdot(v^{\varepsilon}\eta)_{xx}dx.$$

Integrating by parts the first component, applying $v_t^{\varepsilon}(1,t)=0$ and making use of the identity (3.46) we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1|(v^{\varepsilon}\eta)_x|^2dx+\int_0^1\frac{\partial}{\partial x}D^{\alpha}(v^{\varepsilon}\eta)\cdot(v^{\varepsilon}\eta)_{xx}dx=-\int_0^1 f^{\varepsilon}\eta\cdot(v^{\varepsilon}\eta)_{xx}dx+\int_0^1 G(x,t)\cdot(v^{\varepsilon}\eta)_{xx}dx, \tag{3.47}$$

where

$$G:=G_1+G_2:=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x(x-p)^{-\alpha-1}(\eta(x)-\eta(p))v_x^{\varepsilon}(p)dp+\partial^{\alpha}(\eta'\cdot v^{\varepsilon}).$$

By (3.41) we obtain that $\frac{\partial}{\partial x}D^{\alpha}v^{\varepsilon}(\cdot,t)\in AC[0,1]$ for $t>0$. Hence, using $(v^{\varepsilon}\eta)_x(0,t)=0$, Proposition 2.30 and estimate (2.19) we get

$$\int_0^1\frac{\partial}{\partial x}D^{\alpha}(v^{\varepsilon}\eta)\cdot(v^{\varepsilon}\eta)_{xx}dx=\int_0^1 D^{\alpha}(v^{\varepsilon}\eta)_x\cdot(v^{\varepsilon}\eta)_{xx}dx$$
$$=\int_0^1 D^{\alpha}(v^{\varepsilon}\eta)_x\cdot\partial^{1-\alpha}D^{\alpha}(v^{\varepsilon}\eta)_xdx\geq c_{\alpha}\left\|D^{\alpha}(v^{\varepsilon}\eta)_x\right\|^2_{H^{\frac{1-\alpha}{2}}(0,1)}.$$

Applying Proposition 2.32 together with Proposition 2.30 and the fact that $(v^{\varepsilon}\eta)_x$ vanishes at zero we may write

$$\left\|D^{\alpha}(v^{\varepsilon}\eta)_x\right\|^2_{H^{\frac{1-\alpha}{2}}(0,1)}=c_{\alpha}\left\|D^{\frac{\alpha+1}{2}}(v^{\varepsilon}\eta)_x\right\|^2_{L^2(0,1)}=c_{\alpha}\left\|\partial^{\frac{\alpha+1}{2}}(v^{\varepsilon}\eta)_x\right\|^2_{L^2(0,1)}.$$

Finally, using Proposition 2.32 we obtain the estimate

$$\int_0^1\frac{\partial}{\partial x}D^{\alpha}(v^{\varepsilon}\eta)\cdot(v^{\varepsilon}\eta)_{xx}dx\geq c_{\alpha}\left\|(v^{\varepsilon}\eta)_x\right\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}. \tag{3.48}$$

We note that

$$\left|\int_0^1 G(x,t)\cdot(v^{\varepsilon}\eta)_{xx}dx\right|\leq\left\|(v^{\varepsilon}\eta)_{xx}\right\|_{H^{\frac{\alpha-1}{2}}(0,1)}\left\|G(\cdot,t)\right\|_{H^{\frac{1-\alpha}{2}}(0,1)}.$$

Using Remark 2.2 and Young inequality we obtain

$$\left|\int_0^1 G(x,t)\cdot(v^{\varepsilon}\eta)_{xx}dx\right|\leq c(\alpha)\left\|(v^{\varepsilon}\eta)_x\right\|_{H^{\frac{\alpha+1}{2}}(0,1)}\left\|G(\cdot,t)\right\|_{H^{\frac{1-\alpha}{2}}(0,1)}$$
$$\leq\frac{c_{\alpha}}{8}\left\|(v^{\varepsilon}\eta)_x\right\|^2_{H^{\frac{\alpha+1}{2}}(0,1)}+c(\alpha)\left\|G(\cdot,t)\right\|^2_{H^{\frac{1-\alpha}{2}}(0,1)},$$

where c_{α} denotes a constant from estimate (3.48) and by $c(\alpha)$ we denote a generic constant dependent on α. Now we will estimate the $H^{\frac{1-\alpha}{2}}$ - norm of G. Since $(\eta'v^{\varepsilon})(0)=0$, we have

$$\left\|\partial^{\alpha}(\eta'v^{\varepsilon})\right\|_{H^{\frac{1-\alpha}{2}}(0,1)}=\left\|D^{\alpha}(\eta'v^{\varepsilon})\right\|_{H^{\frac{1-\alpha}{2}}(0,1)}\leq c(\alpha)\left\|D^{\frac{1+\alpha}{2}}(\eta'v^{\varepsilon})\right\|_{L^2(0,1)}\leq c(\alpha)\left\|\eta'v^{\varepsilon}\right\|_{H^{\frac{1+\alpha}{2}}(0,1)},$$

where we applied Proposition 2.30 and Proposition 2.32. Thus,

$$\left\|G_2\right\|_{H^{\frac{1-\alpha}{2}}(0,1)}\leq c(\alpha)\left\|\eta'v^{\varepsilon}\right\|_{H^{\frac{1+\alpha}{2}}(0,1)}. \tag{3.49}$$

In order to estimate the $H^{\frac{1-\alpha}{2}}$ - norm of G_1, it is enough to estimate the L^2- norm of $\partial^{\frac{1-\alpha}{2}}G_1$ and apply Proposition 2.32. We note that

$$\Gamma\left(\frac{\alpha+1}{2}\right)\partial^{\frac{1-\alpha}{2}}G_1(x)=\Gamma\left(\frac{\alpha+1}{2}\right)\partial^{\frac{1-\alpha}{2}}\int_0^x(x-p)^{-\alpha-1}(\eta(x)-\eta(p))v_x^{\varepsilon}(p)dp$$

$$= \frac{\partial}{\partial x}\int_0^x (x-p)^{\frac{\alpha-1}{2}}\int_0^p (p-\tau)^{-\alpha-1}(\eta(p)-\eta(\tau))v_x^\varepsilon(\tau)d\tau dp$$

$$= \frac{\partial}{\partial x}\int_0^x v_x^\varepsilon(\tau)\int_\tau^x (x-p)^{\frac{\alpha-1}{2}}(p-\tau)^{-\alpha-1}(\eta(p)-\eta(\tau))dpd\tau = \left\{\begin{array}{l} p=\tau+w(x-\tau)\\ dp=(x-\tau)dw\end{array}\right\}$$

$$= \frac{\partial}{\partial x}\int_0^x v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\frac{1}{w}(\eta(\tau+w(x-\tau))-\eta(\tau))dwd\tau.$$

Using the fact that v_x^ε is bounded with respect to space for each positive time and the estimate

$$|\eta(\tau+w(x-\tau))-\eta(\tau)| \le \|\eta\|_{W^{1,\infty}(0,1)}\, w(x-\tau) \text{ for every } w\in(0,1),\ \ 0\le\tau<x\le 1,$$

it is not difficult to show that we may differentiate under the integral. Furthermore, denoting by $B(\cdot,\cdot)$ the Beta function we may estimate as follows

$$\left|v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\frac{1}{w}(\eta(\tau+w(x-\tau))-\eta(\tau))dx\right|$$
$$\le \|\eta\|_{W^{1,\infty}(0,1)}\, B(1-\alpha,(\alpha+1)/2)\left|v_x^\varepsilon(\tau)(x-\tau)^{1-\frac{\alpha+1}{2}}\right| \to 0 \text{ as } \tau\to x^-.$$

Thus, proceeding with differentiation, we have

$$\Gamma\left(\frac{\alpha+1}{2}\right)\partial^{\frac{1-\alpha}{2}}G_1(x) = \int_0^x v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\eta'(\tau+w(x-\tau))dwd\tau$$
$$-\frac{\alpha+1}{2}\int_0^x v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}-1}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\frac{1}{w}(\eta(\tau+w(x-\tau))-\eta(\tau))dwd\tau.$$

Thus,

$$\|G_1\|_{H^{\frac{1-\alpha}{2}}(0,1)} \le c(\alpha)\left\|\int_0^x v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\eta'(\tau+w(x-\tau))dwd\tau\right\|_{L^2(0,1)}$$
$$+c(\alpha)\left\|\int_0^x v_x^\varepsilon(\tau)(x-\tau)^{-\frac{\alpha+1}{2}}\int_0^1 (1-w)^{\frac{\alpha-1}{2}}w^{-\alpha}\frac{(\eta(\tau+w(x-\tau))-\eta(\tau))}{w(x-\tau)}dwd\tau\right\|_{L^2(0,1)}$$

$$\le c(\alpha)\|\eta\|_{W^{1,\infty}(0,1)}\left\|I^{\frac{1-\alpha}{2}}|v_x^\varepsilon|\right\|_{L^2(0,1)} \le c(\alpha)\|\eta\|_{W^{1,\infty}(0,1)}\|v_x^\varepsilon\|_{H^{\frac{\alpha-1}{2}}(0,1)}, \tag{3.50}$$

where in the last inequality we applied Proposition 2.35. We note that we could skip the absolute value in the last term. Indeed, let us denote by $\langle\cdot,\cdot\rangle$ the duality pairing between the spaces $H^{\frac{\alpha-1}{2}}(0,1)$, $H^{\frac{1-\alpha}{2}}(0,1)$. Since v_x^ε is continuous with respect to space for any positive time, we may write

$$\||v_x^\varepsilon|\|_{H^{\frac{\alpha-1}{2}}(0,1)} = \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}|\langle|v_x^\varepsilon|,w\rangle| = \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}\left|\int_0^1 |v_x^\varepsilon|\cdot w dx\right|$$

$$\le \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}\left|\int_0^1 v_x^\varepsilon\cdot w\cdot\chi_{\{x\in(0,1):v_x^\varepsilon\ge 0\}}dx\right| + \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}\left|\int_0^1 -v_x^\varepsilon\cdot w\cdot\chi_{\{x\in(0,1):v_x^\varepsilon<0\}}dx\right|$$

$$\le \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}\left|\int_0^1 v_x^\varepsilon\cdot wdx\right| + \sup_{w\in H^{\frac{1-\alpha}{2}}(0,1),\|w\|=1}\left|\int_0^1 -v_x^\varepsilon\cdot wdx\right| = 2\|v_x^\varepsilon\|_{H^{\frac{\alpha-1}{2}}(0,1)},$$

where we denoted by χ the characteristic function. Thus, (3.50) is justified. Combining (3.49) and (3.50) we obtain

$$\|G\|^2_{H^{\frac{1-\alpha}{2}}(0,1)} \le c(\alpha,\eta)\|v^\varepsilon\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}. \tag{3.51}$$

By Schwarz inequality, Remark 2.2 and Young inequality we may also write

$$\left|\int_0^1 f^\varepsilon\eta\cdot(v^\varepsilon\eta)_{xx}dx\right| \le \|(v^\varepsilon\eta)_{xx}\|_{H^{\frac{\alpha-1}{2}}(0,1)}\|f^\varepsilon\eta\|_{H^{\frac{1-\alpha}{2}}(0,1)}$$

$$\le \frac{c_\alpha}{8}\|(v^\varepsilon\eta)_x\|^2_{H^{\frac{\alpha+1}{2}}(0,1)} + c(\alpha)\|f^\varepsilon\eta\|^2_{H^{\frac{1-\alpha}{2}}(0,1)}. \tag{3.52}$$

Using estimates (3.48), (3.51) and (3.52) in (3.47) we obtain

$$\frac{d}{dt}\int_0^1|(v^\varepsilon\eta)_x|^2dx + \frac{c_\alpha}{2}\|(v^\varepsilon\eta)_x\|^2_{H^{\frac{\alpha+1}{2}}(0,1)} \le c(\alpha)\|f^\varepsilon\eta\|^2_{H^{\frac{1-\alpha}{2}}(0,1)} + c(\alpha,\eta)\|v^\varepsilon\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}.$$

Applying the estimate (3.45) and recalling that $f^\varepsilon \to f$ in $L^2(0,T;L^1(0,1))$ we obtain for every $t\in(0,T]$

$$\int_0^1|(v^\varepsilon\eta)_x(\cdot,t)|^2dx + \frac{c_\alpha}{2}\int_0^t\|(v^\varepsilon\eta)_x(\cdot,\tau)\|^2_{H^{\frac{\alpha+1}{2}}(0,1)}d\tau$$
$$\le c(\alpha,\eta)\|v_0\|_{H^1(0,1)} + c(\alpha)\|f^\varepsilon\eta\|^2_{L^2(0,t;H^{\frac{1-\alpha}{2}}(0,1))} + c(\alpha,\eta)\|f\|^2_{L^2((0,t);L^1(0,1))}.$$

If we recall that $f^\varepsilon\to f$ in $L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))$, $\eta\equiv 0$ on $[0,(\varepsilon_1+\delta)/2]$ and $\eta\equiv 1$ on $[\delta,1]$ for $0<\varepsilon_1<\delta$, we get

$$\|v^\varepsilon\|^2_{L^\infty(0,T;H^1(\delta,1))} + \frac{c_\alpha}{2}\|v^\varepsilon_x\|^2_{L^2(0,T;H^{\frac{\alpha+1}{2}}(\delta,1))}$$

$$\le c(\alpha,\delta,\varepsilon_1)\left(\|v_0\|^2_{H^1(0,1)} + \|f\|^2_{L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))} + \|f\|^2_{L^2(0,T);L^1(0,1)}\right). \tag{3.53}$$

We obtain that $\{v^\varepsilon\}$ is bounded uniformly with respect to ε in $L^\infty(0,T;H^1(\delta,1))$ and $\{v^\varepsilon_x\}$ is bounded uniformly with respect to ε in $L^2(0,T;H^{\frac{\alpha+1}{2}}(\delta,1))$ for every $\delta\in(\varepsilon_1,1)$. Thus, on a subsequence

$$v^\varepsilon \overset{*}{\rightharpoonup} v \quad\text{in}\quad L^\infty(0,T;H^1(\delta,1)) \quad\text{and}\quad v^\varepsilon_x \rightharpoonup v_x \quad\text{in}\quad L^2(0,T;H^{\frac{\alpha+1}{2}}(\delta,1))$$

for every $\delta\in(\varepsilon_1,1)$. In order to estimate v^ε_t we note that for any $\varepsilon_1<\delta<1$ and any smooth function η such that $\eta\equiv 0$ on $[0,(\varepsilon_1+\delta)/2]$, $\eta\equiv 1$ on $[\delta,1]$ we have

$$v^\varepsilon_t\eta = \partial^\alpha v^\varepsilon_x\eta + f^\varepsilon\eta$$

and by (3.46)

$$\partial^\alpha v^\varepsilon_x\eta = \partial^\alpha(v^\varepsilon\eta)_x + G(t,x) \tag{3.54}$$

Moreover, since $\eta(0)=\eta'(0)=0$, by Proposition 2.32 and Proposition 2.30

$$\|\partial^\alpha(v^\varepsilon\eta)_x\|_{H^{\frac{1-\alpha}{2}}(0,1)} = \left\|D^{\frac{1+\alpha}{2}}(v^\varepsilon\eta)_x\right\|_{L^2(0,1)} \tag{3.55}$$

and by Lemma 3.3

$$\left\|D^{\frac{1+\alpha}{2}}(v^\varepsilon\eta)_x\right\|_{L^2(0,1)} = c(\alpha)\|(v^\varepsilon\eta)_x\|_{H^{\frac{1+\alpha}{2}}(0,1)}. \tag{3.56}$$

Thus, combining (3.51), (3.53), (3.54),(3.55), (3.56), we arrive at

$$\|\partial^\alpha v^\varepsilon_x\|_{L^2(0,T;H^{\frac{1-\alpha}{2}}(\delta,1))} \le c(\alpha,\delta,\varepsilon_1)\left(\|v_0\|^2_{H^1(0,1)} + \|f\|^2_{L^2(0,T;H^{\frac{1-\alpha}{2}}(\varepsilon_1,1))} + \|f\|^2_{L^2(0,T);L^1(0,1)}\right) \tag{3.57}$$

for every $\delta \in (\varepsilon_1, 1)$. Thus, $\partial^\alpha v_x^\varepsilon$ is bounded on $L^2(0,T;H^{\frac{1-\alpha}{2}}(\delta,1))$ for every $\delta \in (\varepsilon_1, 1)$ and so is v_t^ε. It remains to pass to the limit. We fix $\delta_* \in (\varepsilon_1, 1)$. We multiply (3.39) by $w \in L^2(\delta_*, 1)$ and integrate over $(\delta_*, 1)$. Then we multiply by $\Phi \in C_0^\infty(0,T)$ and integrate with respect to time.

$$\int_0^T \Phi \int_{\delta_*}^1 v_t^\varepsilon \cdot w dx dt = \int_0^T \Phi \int_{\delta_*}^1 \frac{\partial}{\partial x} D^\alpha v^\varepsilon \cdot w dx dt + \int_0^T \Phi \int_{\delta_*}^1 f^\varepsilon \cdot w dx dt.$$

Passing to a subsequence we have

$$\int_0^T \Phi \int_{\delta_*}^1 v_t^\varepsilon \cdot w dx dt \to \int_0^T \Phi \int_{\delta_*}^1 v_t \cdot w dx dt$$

and

$$\int_0^T \Phi \int_{\delta_*}^1 f^\varepsilon \cdot w dx dt \to \int_0^T \Phi \int_{\delta_*}^1 f \cdot w dx dt.$$

Moreover, by the identity (3.1) and estimate (3.57) we obtain that there exists $\Upsilon \in L^2(0,T;H^{\frac{1-\alpha}{2}}(\delta_*,1))$ such that on the subsequence

$$\frac{\partial}{\partial x} D^\alpha v^\varepsilon \rightharpoonup \Upsilon \text{ in } L^2(0,T;H^{\frac{1-\alpha}{2}}(\delta_*,1)).$$

Thus, we get that

$$\int_0^T \Phi \int_{\delta_*}^1 \frac{\partial}{\partial x} D^\alpha v^\varepsilon \cdot w dx dt \to \int_0^T \Phi \int_{\delta_*}^1 \Upsilon \cdot w dx dt. \tag{3.58}$$

Let us characterize this limit. We choose $\delta_1 \in (\varepsilon_1, \delta_*)$. Then,

$$\Gamma(1-\alpha)\frac{\partial}{\partial x} D^\alpha v^\varepsilon(x) = \frac{\partial}{\partial x}\int_{\delta_1}^x (x-p)^{-\alpha} v_x^\varepsilon(p) dp - \alpha \int_0^{\delta_1} (x-p)^{-\alpha-1} v_x^\varepsilon(p) dp. \tag{3.59}$$

We will pass to the weak limit with both terms on the r.h.s. separately. Let us begin with the last term in (3.59) If we recall that $\{v_x^\varepsilon\}$ is bounded on $L^2(0,T;H^{\frac{\alpha-1}{2}}(0,1))$, we may write

$$-\alpha \int_0^T \Phi \int_{\delta_*}^1 \int_0^{\delta_1} (x-p)^{-\alpha-1} v_x^\varepsilon(p) dp \cdot w dx dt =$$

$$\int_0^T \Phi \int_{\delta_*}^1 \left\langle \frac{\partial}{\partial x}(x-\cdot)^{-\alpha}, v_x^\varepsilon \right\rangle_{H^{\frac{1-\alpha}{2}}(0,\delta_1)\times H^{\frac{\alpha-1}{2}}(0,\delta_1)} w dx dt$$

$$\to \int_0^T \Phi \int_{\delta_*}^1 \left\langle \frac{\partial}{\partial x}(x-\cdot)^{-\alpha}, v_x \right\rangle_{H^{\frac{1-\alpha}{2}}(0,\delta_1)\times H^{\frac{\alpha-1}{2}}(0,\delta_1)} w dx dt$$

$$= \int_0^T \Phi \int_{\delta_*}^1 \frac{\partial}{\partial x} \left\langle (x-\cdot)^{-\alpha}, v_x \right\rangle_{H^{\frac{1-\alpha}{2}}(0,\delta_1)\times H^{\frac{\alpha-1}{2}}(0,\delta_1)} w dx dt,$$

where the last identity follows from the continuity of the duality pairing. In view of this convergence and identity (3.59) we obtain that on the subsequence $\frac{\partial}{\partial x}\int_{\delta_1}^x (x-p)^{-\alpha} v_x^\varepsilon(p) dp$ is weakly convergent in $L^2(0,T;L^2(\delta_*,1))$.

If we take $\Psi \in C_0^\infty(\delta_*,1)$ we can write

$$\int_0^T \Phi \int_{\delta_*}^1 \frac{\partial}{\partial x}\int_{\delta_1}^x (x-p)^{-\alpha} v_x^\varepsilon(p) dp \Psi dx dt = -\int_0^T \Phi \int_{\delta_*}^1 \int_{\delta_1}^x (x-p)^{-\alpha} v_x^\varepsilon(p) dp \Psi' dx dt.$$

In particular, passing to another subsequence $v_x^\varepsilon \to v$ in $L^2(0,T;L^2(\delta,1))$ for every $\delta > \varepsilon_1$. By continuity of fractional integral on L^2 we obtain that

$$\int_0^T \Phi \int_{\delta_*}^1 \int_{\delta_1}^x (x-p)^{-\alpha} v_x^\varepsilon(p) dp \Psi' dx dt \to \int_0^T \Phi \int_{\delta_*}^1 \int_{\delta_1}^x (x-p)^{-\alpha} v_x(p) dp \Psi' dx dt.$$

Hence, on the subsequence

$$\frac{\partial}{\partial x}\int_{\delta_1}^{x}(x-p)^{-\alpha}v_x^{\varepsilon}(p)dp \rightharpoonup \frac{\partial}{\partial x}\int_{\delta_1}^{x}(x-p)^{-\alpha}v_x(p)dp,$$

where $\frac{\partial}{\partial x}$ is a weak derivative.

Making use of this result together with (3.59) in (3.58) we obtain that, for every $\varepsilon_1 < \delta_1 < \delta_* < 1$

$$\int_0^T \Phi \int_{\delta_*}^1 \frac{\partial}{\partial x} D^{\alpha} v^{\varepsilon} \cdot w dx dt$$

$$\to \int_0^T \Phi \int_{\delta_*}^1 \left[\frac{\partial}{\partial x} I_{\delta_1}^{1-\alpha} v_x + \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\left\langle (x-\cdot)^{-\alpha}, v_x \right\rangle_{H^{\frac{1-\alpha}{2}}(0,\delta_1)\times H^{\frac{\alpha-1}{2}}(0,\delta_1)}\right] w dx dt,$$

which finishes the proof. $\square$

Chapter 4

A space-fractional Stefan problem

In this chapter we will present an example of application of Theorem 3.5. We will solve a space-fractional Stefan problem derived in section 2.4. The results of this chapter, apart from the final section, come from [28].

Our aim is to prove the following theorem.

Theorem 4.1. *Let $b, T > 0$ and $\alpha \in (0,1)$. Let us assume that $u_0 \in H^{1+\alpha}(0,b)$, $u_0' \in {}_0H^{\alpha}(0,b)$, $u_0(b) = 0$ and $u_0 \geq 0$, $u_0 \not\equiv 0$. Further let us assume that there exists $M > 0$ such that for every $x \in [0,b]$*

$$u_0(x) \leq \frac{M\Gamma(2-\alpha)}{b^{1-\alpha}}(b-x).$$

Then, there exists exactly one (u,s) a solution to

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^{\alpha} u = 0 & \textit{in } \{(x,t) : 0 < x < s(t), 0 < t < T\} =: Q_{s,T}, \\ u_x(0,t) = 0, \quad u(t,s(t)) = 0 & \textit{for } t \in (0,T), \\ u(x,0) = u_0(x) & \textit{for } 0 < x < s(0) = b, \\ \dot{s}(t) = -(D^{\alpha}u)(s(t),t) & \textit{for } t \in (0,T), \end{cases} \tag{4.1}$$

such that $s \in C^1[0,T]$, for every $t \in [0,T]$ there holds $0 < \dot{s}(t) \leq M$, $u \in C(\overline{Q_{s,T}})$, $u_t, \frac{\partial}{\partial x} D^{\alpha} u \in C(Q_{s,T})$, $D^{\alpha} u \in C(\overline{Q_{s,T}})$ and for every $t \in [0,T]$ $u_x(\cdot,t) \in {}_0H^{\alpha}(0,s(t))$. Moreover, $u_x \in C(\overline{Q_{s,T}})$ in the case $\alpha \in (\frac{1}{2},1)$, while in the case $\alpha \in (0,\frac{1}{2}]$ we have $u_x \in C(\overline{Q_{s,T}} \setminus (\{t=0\} \times [0,b]))$. Furthermore, the boundary conditions $(4.1)_2$ are satisfied for every $t \in [0,T]$. Finally, there exists $\beta \in (\alpha,1)$, such that for every $t \in (0,T]$ and every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$.

Remark 4.1. *We note that we obtain the continuity of $\dot{s}$ up to the origin because we assume high regularity of the initial condition u_0. Indeed, since $u_{0,x} \in {}_0H^{\alpha}(0,b)$ applying Corollary 2.33 we obtain that $D^{\alpha} u_0 = I^{1-\alpha} u_{0,x} \in {}_0H^1(0,b)$. Hence, we may expect the continuity of $D^{\alpha} u$ up to the initial time.*

Our approach follows the standard methods for solving the classical Stefan problem, presented in [1]. First of all, we focus our attention on the problem considered in a non cylindrical domain with a given function s. We apply a transformation to the cylindrical domain and we find a regular solution by means of the abstract evolution operator theory. Then, we prove the weak extremum principle and the space-fractional version of Hopf lemma, i.e. $(D^\alpha u)(s(t),t) < 0 \ \forall\, t \in (0,T]$. Finally, by the Schauder fixed point theorem, we are able to obtain existence of a pair (u,s) which is a classical solution to (4.1). At last, we prove the monotone dependence upon data in order to obtain the uniqueness of the solution.

4.1. Solution to (4.1) with a given function s.

At first, we will find a regular solution to (4.1) assuming that function s is given. Namely, we will search for a solution to

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } Q_{s,T}, \\ u_x(0,t) = 0, \quad u(t,s(t)) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } 0 < x < b \end{cases} \tag{4.2}$$

with a given function $s : [0,T] \to \mathbb{R}$. We assume that

$$s \in C^{0,1}[0,T], \;\; s(0) = b, \;\; \exists\ M > 0 \text{ such that } 0 < \dot{s}(t) \le M \quad \text{a.e. on} \quad (0,T). \tag{4.3}$$

It is worth to notice that the final result will be proven by the Schauder fixed point theorem in $C[0,T]$. Hence, we do not consider here $s \in C^1[0,T]$, because this space is not closed in $C[0,T]$. We will deduce $C^1[0,T]$ regularity of s at the end of the proof.

We search for a real-valued solution to (4.1), hence henceforth we discuss only real-valued functions.

4.1.1. Transformation to the cylindrical domain

First of all, we will change the coordinates in order to pass to the cylindrical domain. We apply the standard substitution $p = \frac{x}{s(t)}$ and we define

$$v(p,t) := u(s(t)p,t) = u(x,t). \tag{4.4}$$

We will rewrite the system (4.2) in terms of v. Firstly, we note that $\frac{\partial}{\partial p} = s(t)\frac{\partial}{\partial x}$, thus

$$v_p(p,t) = \frac{\partial}{\partial p} v(p,t) = \frac{\partial}{\partial p} u(s(t)p,t) = s(t)\frac{\partial}{\partial x} u(s(t)p,t) = s(t) u_x(x,t),$$

$$v_t(p,t) = \frac{d}{dt} u(s(t)p,t) = u_t(x,t) + p\dot{s}(t) u_x(x,t).$$

Together we have

$$u_t(x,t) = v_t(p,t) - p\frac{\dot{s}(t)}{s(t)} v_p(p,t).$$

Furthermore, since $v_r(r,t) = s(t)u_x(s(t)r,t)$, we may write

$$\Gamma(1-\alpha)(\partial^\alpha v_p)(p,t) = \frac{\partial}{\partial p}\int_0^p (p-r)^{-\alpha} v_r(r,t)dr = s(t)\frac{\partial}{\partial p}\int_0^p (p-r)^{-\alpha} u_x(s(t)r,t)dr$$

$$= \left\{ \begin{array}{l} s(t)r = w \\ s(t)dr = dw \end{array} \right\} = \frac{\partial}{\partial p}\int_0^{s(t)p} (p - \frac{w}{s(t)})^{-\alpha} u_x(w,t)dw$$

$$= s^\alpha(t)\frac{\partial}{\partial p}\int_0^{s(t)p} (s(t)p - w)^{-\alpha} u_x(w,t)dw = s^{\alpha+1}(t)\frac{\partial}{\partial x}\int_0^x (x-w)^{-\alpha} u_x(w,t)dw.$$

In this way we obtained that

$$(\partial^\alpha u_x)(x,t) = \frac{1}{s^{1+\alpha}(t)}(\partial^\alpha v_p)(p,t). \tag{4.5}$$

Denoting

$$v_0(p) = u_0(pb) \tag{4.6}$$

and renaming p by x we obtain that v satisfies

$$\begin{cases} v_t - x\frac{\dot{s}(t)}{s(t)}v_x - \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha v = 0 & \text{for } 0 < x < 1, \ \ 0 < t < T, \\ v_x(0,t) = 0, \quad v(1,t) = 0 & \text{for } 0 < t < T, \\ v(x,0) = v_0(x) & \text{for } 0 < x < 1. \end{cases} \tag{4.7}$$

In the next section we will find a unique solution to (4.7) which will have appropriate regularity properties.

4.1.2. Solution to transformed problem

We will solve the system (4.7) by means of the theory of evolution operators. Let us define the family of operators $A(\cdot) : \mathcal{D}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ given by the following formula

$$A(t) = x\frac{\dot{s}(t)}{s(t)}\frac{\partial}{\partial x} + \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha. \tag{4.8}$$

Let us denote

$$A_1(t) = x\frac{\dot{s}(t)}{s(t)}\frac{\partial}{\partial x} \quad \text{and} \quad A_2(t) = \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha.$$

From Theorem 3.5 and assumption (4.3) we may infer that the family $A_2(\cdot)$ satisfies the assumptions of Theorem 2.15. Indeed, the Theorem 3.5 implies that for every $t \in [0,T]$ $A_2(t)$ is sectorial. Moreover, for $0 \le t, \tau \le T$, $u \in \mathcal{D}_\alpha$

$$\|(A_2(t) - A_2(\tau))u\|_{L^2(0,1)} \le \frac{|s^{1+\alpha}(t) - s^{1+\alpha}(\tau)|}{s^{1+\alpha}(t)s^{1+\alpha}(\tau)}\left\|\frac{\partial}{\partial x}D^\alpha u\right\|_{L^2(0,1)}$$

$$\le \frac{c_\alpha(1+\alpha)M(b+MT)^\alpha}{b^{2(1+\alpha)}}|t-\tau|\,\|u_x\|_{{}_0H^\alpha(0,1)} \le \frac{c_\alpha(1+\alpha)M(b+MT)^\alpha}{b^{2(1+\alpha)}}|t-\tau|\,\|u\|_{\mathcal{D}_\alpha},$$

where we applied identity (3.1), Proposition (2.32) and the assumption (4.3). In consequence, we obtain that

$$t \mapsto A_2(t) \in C^{0,1}([0,T]; B(\mathcal{D}_\alpha, L^2(0,1))). \tag{4.9}$$

However, since $\dot{s}$ is not Hölder continuous we are not allowed to use directly the results from Theorem 2.15 to the family $A(\cdot)$. Hence, we are going to find firstly a mild solution to the problem (4.7). Then we will show that this mild solution actually satisfies (4.7) almost everywhere. Finally, we will further increase the regularity of the solution. Let us denote by $\{G(t,\sigma) : 0 \le \sigma \le t \le T\}$ the evolution operator associated with $A_2(t)$, given by Theorem 2.15. For clarity we rewrite here, the general result from Proposition 2.18 in our special case. If $g \in [L^2(0,1), \mathcal{D}_\alpha]_\delta$ then for any $0 \le \sigma < t \le T$ and every $\delta \in (0,1)$

$$\|G(t,\sigma)g\|_{\mathcal{D}_\alpha} \le \frac{c}{(t-\sigma)^{1-\delta}} \|g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta} . \tag{4.10}$$

Moreover, for any $0 \le \delta < \theta < 1$, we have

$$\|G(t,\sigma)g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\theta} \le \frac{c}{(t-\sigma)^{\theta-\delta}} \|g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta} \tag{4.11}$$

and for $\theta \in (0,1)$, $\delta \in (0,1]$

$$\|A_2(t)G(t,\sigma)g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\theta} \le \frac{c}{(t-\sigma)^{1+\theta-\delta}} \|g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta} . \tag{4.12}$$

Finally, for every $a \in (0,1)$ and every $0 \le \sigma < r < t \le T$

$$\|A_2(t)G(t,\sigma)g - A_2(r)G(r,\sigma)g\|_{L^2(0,1)}$$

$$\le c\left(\frac{(t-r)^a}{(r-\sigma)^{1-\delta}} + \frac{1}{(r-\sigma)^{1-\delta}} - \frac{1}{(t-\sigma)^{1-\delta}}\right) \|g\|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta} . \tag{4.13}$$

The constant c in estimates above is positive and depends only on $\alpha, \theta, \delta, T$ and b, M from (4.3). Moreover, function $T \mapsto c(\alpha, \theta, \delta, b, M, T)$ is increasing.
We would like to find a mild solution to (4.7). For this purpose we rewrite this equation in the integral form

$$v(x,t) = G(t,0)v_0(x) + \int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)} x v_x(x,\sigma) d\sigma. \tag{4.14}$$

We say that $v \in C([0,T]; \mathcal{D}_\alpha)$ is a mild solution to (4.7) if it satisfies (4.14).

Theorem 4.2. *Let us assume that $v_0 \in \mathcal{D}_\alpha$. Then, there exists a unique solution to (4.14) belonging to $C([0,T]; \mathcal{D}_\alpha)$.*

Proof. We will prove this result by the Banach fixed point theorem. We define the operator

$$(Pv)(x,t) = G(t,0)v_0(x) + \int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)} x v_x(x,\sigma) d\sigma. \tag{4.15}$$

We will show that $P : C([0,T]; \mathcal{D}_\alpha) \to C([0,T]; \mathcal{D}_\alpha)$. Indeed, let $v \in C([0,T]; \mathcal{D}_\alpha)$. Since $v_0 \in \mathcal{D}_\alpha$ by Proposition 2.16 we obtain that $G(t,0)v_0 \in C([0,T]; \mathcal{D}_\alpha)$. Let us pass to the second term. We will prove that

$$A_2(t)\int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)} x v_x(x,\sigma) d\sigma \in C([0,T]; L^2(0,1)). \tag{4.16}$$

We note that, since $v \in C([0,T];\mathcal{D}_\alpha)$, we have $v_x \in C([0,T];{}_0H^\alpha(0,1))$, hence $xv_x \in C([0,T];{}_0H^\alpha(0,1))$. It is worth to notice that $v_x(1,t)$ does not have to vanish, thus we cannot consider xv_x as an element of $C([0,T];[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\alpha}{\alpha+1}})$ for $\alpha \geq \frac{1}{2}$. That is way, we proceed as follows. Let us denote $\delta = \frac{\alpha}{\alpha+1}$ in the case $\alpha < \frac{1}{2}$ and for $\alpha \in [\frac{1}{2},1)$ let us mean by δ an arbitrary number belonging to the interval $(0,\frac{1}{2(1+\alpha)})$. Then, in view of characterization (3.40) we obtain that

$$xv_x \in C([0,T];[L^2(0,1),\mathcal{D}_\alpha]_\delta). \tag{4.17}$$

Let us denote

$$f(x,\sigma) := \frac{\dot{s}(\sigma)}{s(\sigma)} xv_x(x,\sigma). \tag{4.18}$$

Since for every $t \in [0,T]$ the operator $A_2(t)$ is sectorial, then in particular it is closed and by [6, Proposition C.4] we may pass with $A_2(t)$ under the integral sign. Hence, for any $0 \leq \tau < t \leq T$ we may estimate as follows

$$\left\| A_2(t)\int_0^t G(t,\sigma)f(\cdot,\sigma)d\sigma - A_2(\tau)\int_0^\tau G(\tau,\sigma)f(\cdot,\sigma)d\sigma \right\|_{L^2(0,1)}$$

$$= \left\| \int_0^t A_2(t)G(t,\sigma)f(\cdot,\sigma)d\sigma - \int_0^\tau A_2(\tau)G(\tau,\sigma)f(\cdot,\sigma)d\sigma \right\|_{L^2(0,1)} \leq$$

$$\int_\tau^t \|A_2(t)G(t,\sigma)f(\cdot,\sigma)\|_{L^2(0,1)}\, d\sigma + \int_0^\tau \|(A_2(t)G(t,\sigma) - A_2(\tau)G(\tau,\sigma))f(\cdot,\sigma)\|_{L^2(0,1)}\, d\sigma \equiv: J_1.$$

We note that for any $w \in \mathcal{D}_\alpha$, using assumption (4.3), identity (3.1) and Proposition 2.32, we have

$$\|A_2(t)w\|_{L^2(0,1)} \leq \frac{1}{b^{1+\alpha}} \|\partial^\alpha w_x\|_{L^2(0,1)} \leq \frac{c_\alpha}{b^{1+\alpha}} \|w_x\|_{{}_0H^\alpha(0,1)} \leq \frac{c_\alpha}{b^{1+\alpha}} \|w\|_{\mathcal{D}_\alpha}. \tag{4.19}$$

Hence, making use of (4.10) and (4.13) we may estimate J_1 as follows,

$$|J_1| \leq c\,\|f\|_{L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)} \int_\tau^t (t-\sigma)^{\delta-1} d\sigma$$

$$+c\,\|f\|_{L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)} \int_0^\tau \frac{(t-\tau)^a}{(\tau-\sigma)^{1-\delta}} + \frac{1}{(\tau-\sigma)^{1-\delta}} - \frac{1}{(t-\sigma)^{1-\delta}} d\sigma$$

$$\leq \frac{c}{\delta}\|f\|_{L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)} \left(2(t-\tau)^\delta + (t-\tau)^a\tau^\delta + \tau^\delta - t^\delta\right)$$

for any $a \in (0,1)$. The expression above tends to zero as $\tau \to t$ for any $0 \leq \tau < t \leq T$, hence (4.16) is proven. We note that for any $w \in \mathcal{D}_\alpha$, since $w(1) = 0$ we may apply the Poincaré inequality to obtain

$$\|w\|^2_{\mathcal{D}_\alpha} = \|w\|^2_{L^2(0,1)} + \|w_x\|^2_{{}_0H^\alpha(0,1)} \leq c\,\|w_x\|^2_{{}_0H^\alpha(0,1)} \leq c_\alpha \left\| \frac{\partial}{\partial x} D^\alpha w \right\|^2_{L^2}.$$

Hence, we observe that (4.3) together with (4.16) leads to

$$\int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)} xv_x(x,\sigma)d\sigma \in C([0,T];\mathcal{D}_\alpha). \tag{4.20}$$

Thus, we have shown that $P : C([0,T];\mathcal{D}_\alpha) \to C([0,T];\mathcal{D}_\alpha)$.

Now we will show that P is a contraction on $C([0,T_1];\mathcal{D}_\alpha)$ for T_1 small enough. To this end we fix $v,w \in C([0,T_1];\mathcal{D}_\alpha)$. Then, we may estimate using (4.3), (4.10) and (4.17)

$$\|Pv - Pw\|_{C([0,T_1];\mathcal{D}_\alpha)} \leq \sup_{t\in(0,T_1)} \frac{M}{b}\int_0^t \|G(t,\sigma)x[v_x - w_x](\cdot,\sigma)\|_{\mathcal{D}_\alpha}\, d\sigma$$

$$\leq \frac{cM}{b} \sup_{t\in(0,T_1)} \int_0^t (t-\sigma)^{\delta-1} \|[v_x - w_x](\cdot,\sigma)\|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta}\, d\sigma$$

$$\leq \frac{cM}{b} \sup_{t\in(0,T_1)} \int_0^t (t-\sigma)^{\delta-1} d\sigma\, \|v_x - w_x\|_{C([0,T_1];{}_0H^\alpha(0,1))} \leq \frac{cM}{b}\frac{T_1^\delta}{\delta} \|v-w\|_{C([0,T_1];\mathcal{D}_\alpha)}\,.$$

Hence, for $T_1 < \left(\frac{b\delta}{cM}\right)^{\frac{1}{\delta}}$ the operator P is a contraction on $C([0,T_1];\mathcal{D}_\alpha)$. Thus, by the Banach fixed point theorem, we obtain the existence of a unique solution to (4.14) on the interval $[0,T_1]$ which belongs to $C([0,T_1];\mathcal{D}_\alpha)$. In order to extend the solution to the whole interval $[0,T]$ we assume that we have already obtained the solution $\tilde{v}$ to (4.14) on the interval $[0,T_k]$ for fixed $k \in \mathbb{N}\setminus\{0\}$. We will find a unique solution on the interval $[0,T_{k+1}]$, where $T_{k+1} > T_k$. We define the space

$$X_k(T_{k+1}) = \{v \in C([0,T_{k+1}];\mathcal{D}_\alpha) : v \equiv \tilde{v} \text{ on } [0,T_k]\},$$

with a norm induced from $C([0,T_{k+1}];\mathcal{D}_\alpha)$. Then, by definition of $\tilde{v}$ and the same reasoning as above we obtain that the operator P defined by (4.15) satisfy $P : X_k(T_{k+1}) \to X_k(T_{k+1})$. Furthermore, for $v^1, v^2 \in X_k(T_{k+1})$ there holds

$$\left\|Pv^1 - Pv^2\right\|_{C([0,T_{k+1}];\mathcal{D}_\alpha)} = \left\|\int_{T_k}^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)}x[v_x^1 - v_x^2](\cdot,\sigma)d\sigma\right\|_{C([T_k,T_{k+1}];\mathcal{D}_\alpha)}.$$

Applying (4.3) and estimate (4.10) we get

$$\left\|Pv^1 - Pv^2\right\|_{C([0,T_{k+1}];\mathcal{D}_\alpha)} \leq \sup_{t\in(T_k,T_{k+1})} \frac{M}{b}\int_{T_k}^t \left\|G(t,\sigma)x[v_x^1 - v_x^2](\cdot,\sigma)\right\|_{\mathcal{D}_\alpha} d\sigma$$

$$\leq \frac{cM}{b} \sup_{t\in(T_k,T_{k+1})} \int_{T_k}^t (t-\sigma)^{\delta-1} d\sigma \left\|v_x^1 - v_x^2\right\|_{C([0,T_{k+1}];[L^2(0,1);\mathcal{D}_\alpha]_\delta)}$$

$$\leq \frac{cM}{b}\frac{(T_{k+1}-T_k)^\delta}{\delta}\left\|v^1 - v^2\right\|_{C([0,T_{k+1}];\mathcal{D}_\alpha)}.$$

Hence, for $(T_{k+1}-T_k) < \left(\frac{b\delta}{cM}\right)^{\frac{1}{\delta}}$ the operator P is a contraction on $X_k(T_{k+1})$ and we may extend uniquely the solution $\tilde{v}$ on the interval $[0,T_{k+1}]$. The length of the interval $[T_k,T_{k+1}]$ does not depend on k. Thus, after a finite number of steps we obtain the unique solution to (4.14) which belongs to $C([0,T];\mathcal{D}_\alpha)$. □

Lemma 4.3. *The mild solution v obtained in Theorem 4.2 satisfies $v \in C([0,T];\mathcal{D}_\alpha)$, $v_t \in L^\infty(0,T;L^2(0,1))$ and*

$$v_t - A(t)v = 0$$

for almost all $t \in [0,T]$ in the sense of $L^2(0,1)$.

Proof. Using definition (4.18) we may rewrite (4.14) as follows

$$v(x,t) = G(t,0)v_0(x) + \int_0^t G(t,\sigma)f(x,\sigma)d\sigma. \tag{4.21}$$

The proof is based on the reasoning carried in the proof of [19, Lemma 6.2.1]. We note that, since f is not Hölder continuous, we can not apply [19, Lemma 6.2.1] directly. We will show that v which satisfies (4.21) is differentiable. By Proposition 2.16 for every $t \in [0,T]$ we have

$$\frac{\partial}{\partial t}G(t,0)v_0 = A_2(t)G(t,0)v_0 \text{ in } L^2(0,1).$$

We will calculate the difference quotient of the second term on the right hand side of (4.21). Let us assume that $h > 0$, in the case $h < 0$ the proof is similar. We have

$$\frac{1}{h}\left[\int_0^{t+h} G(t+h,\sigma)f(x,\sigma)d\sigma - \int_0^t G(t,\sigma)f(x,\sigma)d\sigma\right]$$

$$= \frac{1}{h}\int_0^t (G(t+h,\sigma) - G(t,\sigma))f(x,\sigma)d\sigma + \frac{1}{h}\int_t^{t+h} G(t+h,\sigma)f(x,\sigma)d\sigma =: I_1 + I_2.$$

In order to deal with I_1 we recall that by Definition 2.15 for every $0 \le \sigma < t \le T$ and every $g \in L^2(0,1)$ the following limit holds in $L^2(0,1)$

$$\lim_{h\to 0}\frac{1}{h}(G(t+h,\sigma) - G(t,\sigma))g = A_2(t)G(t,\sigma)g.$$

Making use of (4.17) we obtain that $f \in L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)$, where $\delta = \frac{\alpha}{1+\alpha}$ for $\alpha \in (0,\frac{1}{2})$ and δ denotes any fixed number from the interval $(0,\frac{1}{2(1+\alpha)})$ if $\alpha \in [\frac{1}{2},1)$. Further, we note that

$$\left\|\frac{1}{h}[G(t+h,\sigma) - G(t,\sigma)]f(\cdot,\sigma)\right\|_{L^2(0,1)} = \left\|\frac{1}{h}\int_t^{t+h}\frac{\partial}{\partial p}G(p,\sigma)f(\cdot,\sigma)d\sigma\right\|_{L^2(0,1)}$$

$$= \left\|\frac{1}{h}\int_t^{t+h} A(p)G(p,\sigma)f(\cdot,\sigma)dp\right\|_{L^2(0,1)} \le \frac{c}{h}\int_t^{t+h}(p-\sigma)^{\delta-1}dp\,\|f\|_{L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)}$$

$$\le c(t-\sigma)^{\delta-1}\|f\|_{L^\infty(0,T;[L^2(0,1),\mathcal{D}_\alpha]_\delta)},$$

where we applied (4.10) and (4.19). Hence, we may apply the Lebesgue dominated convergence theorem to pass to the limit under the integral sign in I_1 and we get

$$\frac{1}{h}\int_0^t (G(t+h,\sigma) - G(t,\sigma))f(x,\sigma)d\sigma \to \int_0^t A_2(t)G(t,\sigma)f(x,\sigma)d\sigma.$$

We decompose I_2 as follows

$$\frac{1}{h}\int_t^{t+h} G(t+h,\sigma)f(x,\sigma)d\sigma = \frac{1}{h}\int_t^{t+h} G(t,\sigma)f(x,\sigma)d\sigma$$

$$+\frac{1}{h}\int_t^{t+h}(G(t+h,\sigma) - G(t,\sigma))f(x,\sigma)d\sigma = I_{2,1} + I_{2,2}.$$

We note that due to the Lebesgue differentiation theorem in Banach spaces (see [4]) we obtain that $I_{2,1}$ converges to $f(x,t)$ in $L^2(0,1)$ for almost all $t \in (0,T]$. For $I_{2,2}$ we have

$$I_{2,2} = \frac{1}{h}\int_t^{t+h}(G(t+h,t) - E)G(t,\sigma)f(x,\sigma)d\sigma = \frac{(G(t+h,t) - E)}{h}\int_t^{t+h} G(t,\sigma)f(x,\sigma)d\sigma.$$

Thus, using again the Lebesgue differentiation theorem in Banach spaces and the continuity of $G(t,\cdot)$ in $L^2(0,1)$ we obtain that $I_{2,2}$ converges to zero in $L^2(0,1)$ for almost all $t \in (0,T]$. Summing up the results we obtain that the following identity holds in $L^2(0,1)$ for almost all $t \in [0,T]$

$$v_t(x,t) = A_2(t)G(t,0)v_0(x) + A_2(t)\int_0^t G(t,\sigma)f(x,\sigma)d\sigma + f(x,t).$$

Applying formula (4.21) and the definitions of f and A_2 we get that

$$v_t(x,t) = \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha v(x,t) + \frac{\dot{s}(t)}{s(t)}xv_x(x,t)$$

for almost all $t \in [0,T]$ in $L^2(0,1)$ and we obtain the claim of lemma. □

Our aim is to obtain a solution to (4.2) regular enough to satisfy the weak extremum principle. As it will be seen in the final section, our solution u has to fulfill the following: there exists $\beta \in (\alpha,1)$ such that

$$\text{for every } t \in (0,T) \text{ and every } 0 < \varepsilon < \omega < s(t) \;\; u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega). \tag{4.22}$$

Thus, we need to increase the space regularity of the transformed problem (4.7). The main difficulty is that, from what we have proved by now, $v_x \in {}_0H^\alpha(0,1)$ but v_x need not vanish at the right endpoint of the interval. Hence, we are allowed to consider v_x as an element of the interpolation space $[L^2(0,1),\mathcal{D}_\alpha]_\delta$ only for δ smaller than $\frac{1}{2(1+\alpha)}$. However, in order to obtain higher regularity, we have to examine the behaviour of $A_2(t)G(t,\sigma)f(x,\sigma)$ more carefully. The next lemma establishes the regularity properties of an evolution operator $G(t,\sigma)$ acting on the elements of $H^a(0,1)$ for $a > \frac{1}{2}$. At first we will discuss the case $\alpha \in (\frac{1}{2},1)$. Then, we will present more technical result in the case $\alpha \in (0,\frac{1}{2}]$.

Lemma 4.4. *Let us assume that $\alpha \in (\frac{1}{2},1)$ and $u_\sigma \in {}_0H^\alpha(0,1)$. We denote by u the solution to the equation*

$$\begin{cases} u_t = A_2(t)u & \text{for } \; 0<x<1, \;\; 0 \le \sigma < t < T, \\ u(x,\sigma) = u_\sigma(x) & \text{for } \; 0<x<1, \end{cases} \tag{4.23}$$

given by the evolution operator generated by the family $A_2(t)$. Then, for every $0 < \gamma < \alpha$, for every $0 < \varepsilon < \omega < 1$ there exists a positive constant $c = c(\alpha,b,M,T,\varepsilon,\omega,\gamma)$, where b, M comes from (4.3), such that for every $t \in (\sigma,T]$ there holds

$$\|A_2(t)u(\cdot,t)\|_{H^\gamma(\varepsilon,\omega)} \le c(t-\sigma)^{-\frac{1+\gamma}{1+\alpha}} \|u_\sigma\|_{{}_0H^\alpha(0,1)}.$$

Proof. We note that since $\alpha > \frac{1}{2}$ in view of characterization (3.40) we have $u_\sigma \in [L^2(0,1),\mathcal{D}_\alpha]_\nu$ for every $\nu \in (0,\frac{1}{2(1+\alpha)})$. Hence, by Theorem 3.6 $u \in C([\sigma,T];L^2(0,1)) \cap C((\sigma,T];\mathcal{D}_\alpha) \cap C^1((\sigma,T];L^2(0,1))$ and by (4.10)

$$\|u(\cdot,t)\|_{\mathcal{D}_\alpha} \le c(t-\sigma)^{\nu-1} \|u_\sigma\|_{[L^2(0,1),\mathcal{D}_\alpha]_\nu}. \tag{4.24}$$

We recall that the interpolation constant c depends on the parameters of interpolation as well as on α, T and b, M from (4.3). However, here and henceforth we neglect this dependency in notation and leave it just in the final results. Moreover, we note that the constant $c > 0$ may change from line to line. We fix $0 < \varepsilon < \omega < 1$ and we set $\omega_* = \frac{1+\omega}{2}$. Let us discuss the approximate problem. We choose a sequence $\{\varphi^k\}$ such that

$$\{\varphi^k\} \subseteq \mathcal{D}_\alpha, \;\; \varphi^k \to u_\sigma \text{ in } {}_0H^\alpha(0,\omega_*) \text{ and } \varphi^k \to u_\sigma \text{ in } H^{\bar\gamma}(0,1) \text{ for every } \bar\gamma < \frac{1}{2}. \tag{4.25}$$

Let us justify that such approximate sequence exists. We take a sequence $\psi_k \in {}_0C^\infty(0,1)$ such that $\psi_k \to u_\sigma$ in ${}_0H^\alpha(0,1)$. Then we define $\varphi^k = \psi_k - \psi_k(1)\varrho_k$, where ϱ_k is a sequence of smooth, non-decreasing functions such that $\operatorname{supp}\varrho_k \subseteq [1-\frac{1}{k},1]$, $\varrho_k(1) = 1$ and $|\varrho_k'| \leq 2k$. Then, $\varphi^k \to u_\sigma$ in ${}_0H^\alpha(0,\omega_*)$ and $\varphi^k \in \mathcal{D}_\alpha$. We will show that $\varphi^k \to u_\sigma$ in $H^{\bar\gamma}(0,1)$ for every $\bar\gamma < \frac{1}{2}$. We note that since $\psi_k \to u_\sigma$ in ${}_0H^\alpha(0,1)$ we have $\psi_k(1) \to u_\sigma(1)$. Thus, it is enough to show that $\varrho_k \to 0$ in $H^{\bar\gamma}(0,1)$ for every $\bar\gamma < \frac{1}{2}$. The convergence in $L^2(0,1)$ is straightforward. Moreover, we may calculate

$$\int_{1-\frac{1}{k}}^{1}\int_{1-\frac{1}{k}}^{x} \frac{|\varrho_k(x)-\varrho_k(y)|^2}{|x-y|^{1+2\bar\gamma}}dydx \leq 2k^2\int_{1-\frac{1}{k}}^{1}\int_{1-\frac{1}{k}}^{x}|x-y|^{1-2\bar\gamma}dydx$$

$$= \frac{k^2}{1-\bar\gamma}\int_{1-\frac{1}{k}}^{1}|x-(1-\frac{1}{k})|^{2(1-\bar\gamma)}dx = \frac{k^{2\bar\gamma-1}}{(1-\bar\gamma)(1+2(1-\bar\gamma))} \to 0 \text{ as } k\to\infty.$$

Hence, the existence of the sequence in (4.25) is justified. Recalling the characterization (3.40), applying (4.10) and (4.12), we obtain that the solution to

$$\begin{cases} u^k_t = A_2(t)u^k & \text{for } \;\; 0<x<1, \;\; 0\leq\sigma<t<T, \\ u^k(x,\sigma) = \varphi^k(x) & \text{for } \;\; 0<x<1, \end{cases} \tag{4.26}$$

satisfies for every $0 \leq \bar\gamma < \bar\gamma_1 < \frac{1}{2}$

$$\left\|A_2(t)(u-u^k)(\cdot,t)\right\|_{H^{\bar\gamma}(0,1)} \leq c(t-\sigma)^{-1+\frac{\bar\gamma_1-\bar\gamma}{1+\alpha}}\left\|u_\sigma - \varphi^k\right\|_{H^{\bar\gamma_1}(0,1)}. \tag{4.27}$$

Hence, for every $0 \leq \bar\gamma < \frac{1}{2}$ and every $t \in (\sigma, T]$

$$\frac{\partial}{\partial x}D^\alpha u^k \to \frac{\partial}{\partial x}D^\alpha u \;\; \text{ in } \;\; H^{\bar\gamma}(0,1). \tag{4.28}$$

Furthermore, for k large enough and every $0 \leq \bar\gamma < \bar\gamma_1 < \frac{1}{2}$ we have

$$\left\|A_2(t)u^k(\cdot,t)\right\|_{H^{\bar\gamma}(0,1)} \leq c(t-\sigma)^{-1+\frac{\bar\gamma_1-\bar\gamma}{1+\alpha}}\left\|u_\sigma\right\|_{H^{\bar\gamma_1}(0,1)}. \tag{4.29}$$

We will prove a uniform bound of the sequence $\{u^k\}$ in more regular spaces locally on $(0,1)$. To this end, we introduce a smooth nonnegative cut-off function η such that $\eta \equiv 0$ on $[0,\frac{\varepsilon}{2}] \cup [\omega_*,1]$, $\eta \equiv 1$ on $[\varepsilon,\omega]$. Making use of the regularity of the sequence $\{u^k\}$ we note that we may apply the operator ∂^α to (4.26). Then we multiply the result by η. Applying Proposition 2.26 we arrive at

$$\eta\partial^\alpha\frac{\partial}{\partial x}D^\alpha u^k = -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}(\eta(x)-\eta(p))\frac{\partial}{\partial x}D^\alpha u^k(p)dp + \partial^\alpha(\frac{\partial}{\partial x}D^\alpha u^k\cdot\eta)$$

$$= -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}(\eta(x)-\eta(p))\frac{\partial}{\partial x}D^\alpha u^k(p)dp + \partial^\alpha\frac{\partial}{\partial x}(\eta D^\alpha u^k) - \partial^\alpha(\eta' D^\alpha u^k).$$

From Remark 2.6 we have

$$\partial^\alpha \frac{\partial}{\partial x}(\eta D^\alpha u^k)(x,t) = \partial^\alpha \frac{\partial}{\partial x}(\eta(\partial^\alpha u^k(x,t) - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}u^k(0,t)))$$
$$= \partial^\alpha \frac{\partial}{\partial x}(\eta \cdot \partial^\alpha u^k)(x,t) + \tilde{\eta}(x)u^k(0,t),$$

where we denoted

$$\tilde{\eta} := -\frac{1}{\Gamma(1-\alpha)}\partial^\alpha \frac{\partial}{\partial x}(\eta \cdot x^{-\alpha}).$$

We note that since $\eta \equiv 0$ near zero function $\tilde{\eta}$ is smooth. In view of identity (3.1) we obtain that $\{u^k\}$ satisfy the system of equations

$$\begin{cases} (\partial^\alpha u^k \cdot \eta)_t - A_2(t)(\partial^\alpha u^k \cdot \eta) = F^k & \text{for } 0 < x < 1,\ \ 0 \le \sigma < t < T, \\ (\partial^\alpha u^k \cdot \eta)(\cdot,\sigma) = \partial^\alpha \varphi^k \cdot \eta & \text{for } 0 < x < 1, \end{cases} \tag{4.30}$$

where

$$F^k := \frac{1}{s^{1+\alpha}(t)}\left[\frac{-\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}(\eta(x)-\eta(p))\frac{\partial}{\partial x}D^\alpha u^k(p)dp - \partial^\alpha(\eta' D^\alpha u^k) + \tilde{\eta}u^k(0,t)\right].$$

Let us show a uniform estimate on the L^2 - norm of F^k. At first, by the Sobolev embedding we have

$$\left\|\tilde{\eta}u^k(0,t)\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|u^k(\cdot,t)\right\|_{C([0,1])} \le c(\varepsilon,\omega)\left\|A_2(t)u^k(\cdot,t)\right\|_{L^2(0,1)}. \tag{4.31}$$

Furthermore, using (4.11) we may estimate more precisely that for any $0 < \gamma < \frac{1}{2} < \beta < 1$

$$\left|u^k(0,t)\right| \le c\left\|u^k(\cdot,t)\right\|_{{}_0H^\beta(0,1)} \le c(t-\sigma)^{\frac{\gamma-\beta}{1+\alpha}}\left\|\varphi^k\right\|_{H^\gamma(0,1)}. \tag{4.32}$$

We note that for every $x, p \in [0,1]$, $x \ne p$ we have

$$\left|\frac{\eta(x)-\eta(p)}{x-p}\right| \le \|\eta\|_{W^{1,\infty}(0,1)},$$

hence,

$$\frac{1}{\Gamma(1-\alpha)}\left|\int_0^x (x-p)^{-\alpha-1}(\eta(x)-\eta(p))A_2(t)u^k(p)dp\right| \le \|\eta\|_{W^{1,\infty}(0,1)} I^{1-\alpha}\left|A_2(t)u^k(x,t)\right|.$$

Since $I^{1-\alpha}$ is bounded on $L^2(0,1)$ we obtain that

$$\left\|\frac{\alpha}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha-1}(\eta(x)-\eta(p))A_2(t)u^k(p)dp\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|A_2(t)u^k(\cdot,t)\right\|_{L^2(0,1)}.$$

By Proposition 2.32 we may write

$$\frac{1}{s^{1+\alpha}(t)}\left\|\partial^\alpha(\eta' D^\alpha u^k)\right\|_{L^2(0,1)} \le \frac{c_\alpha}{s^{1+\alpha}(t)}\left\|(\eta' D^\alpha u^k)\right\|_{{}_0H^\alpha(0,1)} \le \frac{c}{s^{1+\alpha}(t)}\left\|(\eta' D^\alpha u^k)\right\|_{{}_0H^1(0,1)}$$
$$\le \frac{c}{s^{1+\alpha}(t)}\left\|(\eta'' D^\alpha u^k)\right\|_{L^2(0,1)} + \frac{c}{s^{1+\alpha}(t)}\left\|(\eta' \frac{\partial}{\partial x}D^\alpha u^k)\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|A_2(t)u^k(\cdot,t)\right\|_{L^2(0,1)}.$$

Combining (4.31), the last two estimates and (4.29) we obtain that for every $0 < \bar{\gamma} < \frac{1}{2}$

$$\left\|F^k(\cdot,t)\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{\frac{\bar{\gamma}}{(1+\alpha)}-1}\|u_\sigma\|_{H^{\bar{\gamma}}(0,1)}. \tag{4.33}$$

We will show that for every $k \in \mathbb{N}$ functions $\partial^\alpha u^k \cdot \eta$ and F^k satisfy the assumptions of Proposition 2.17. By Proposition 2.16 we have $u^k \in C([\sigma,T];\mathcal{D}_\alpha)$. This, together with the continuity of function s and estimates above leads to $F^k \in C((\sigma,T];L^2(0,1)) \cap L^1(\sigma,T;L^2(0,1))$. Due to Remark 2.6 $(\partial^\alpha u^k \cdot \eta)(x,t) = (D^\alpha u^k \cdot \eta)(x,t) + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} u^k(0,t) \cdot \eta(x)$ and the last component belongs to $C([\sigma,T];L^2(0,1)) \cap C((\sigma,T];\mathcal{D}_\alpha)$. Thus, it is enough to show that $D^\alpha u^k \cdot \eta \in C([\sigma,T];L^2(0,1)) \cap C((\sigma,T];\mathcal{D}_\alpha)$. At first, since $u^k \in C([\sigma,T];\mathcal{D}_\alpha)$, we have $D^\alpha u^k \cdot \eta \in C([\sigma,T];L^2(0,1))$ (actually it belongs to $C([\sigma,T];{}_0H^1(0,1))$). Let us show that $D^\alpha u^k \cdot \eta \in C((\sigma,T];\mathcal{D}_\alpha)$. We note that for any $0 < \beta < 1+\alpha$

$$\left\| \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,t) \right\|_{H^\beta(0,1)} \leq \left\| \eta' D^\alpha u^k(\cdot,t) \right\|_{H^\beta(0,1)} + \left\| \eta \frac{\partial}{\partial x} D^\alpha u^k(\cdot,t) \right\|_{H^\beta(0,1)}.$$

Applying estimate (4.12) we obtain that

$$\left\| \frac{\partial}{\partial x} D^\alpha u^k(\cdot,t) \right\|_{H^\beta(0,1)} \leq \left\| \frac{\partial}{\partial x} D^\alpha u^k(\cdot,t) \right\|_{[L^2(0,1),\mathcal{D}_\alpha]_{\frac{\beta}{\alpha+1}}} \leq c(t-\sigma)^{-\frac{\beta}{1+\alpha}} \left\| \varphi^k \right\|_{\mathcal{D}_\alpha} \tag{4.34}$$

hence, using the fact that η vanishes near zero, we have

$$\frac{\partial}{\partial x}(\eta D^\alpha u^k) \in L^\infty_{loc}((\sigma,T];{}_0H^\beta(0,1)) \text{ for every } 0 < \beta < \alpha+1. \tag{4.35}$$

Moreover for every $\sigma < \tau, t \leq T$

$$\left\| \eta D^\alpha u^k(\cdot,t) - \eta D^\alpha u^k(\cdot,\tau) \right\|^2_{\mathcal{D}_\alpha}$$

$$= \left\| \eta D^\alpha u^k(\cdot,t) - \eta D^\alpha u^k(\cdot,\tau) \right\|^2_{L^2(0,1)} + \left\| \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,t) - \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,\tau) \right\|^2_{{}_0H^\alpha(0,1)}.$$

We have already shown that the first norm tends to zero as $\tau \to t$. We apply the interpolation estimate ([19, Corollary 1.2.7.]) to the second term. Then, we have

$$\left\| \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,t) - \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,\tau) \right\|^2_{{}_0H^\alpha(0,1)}$$

$$\leq c \left\| \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,t) - \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,\tau) \right\|^{\frac{\alpha}{\beta}}_{{}_0H^\beta(0,1)} \left\| \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,t) - \frac{\partial}{\partial x}(\eta D^\alpha u^k)(\cdot,\tau) \right\|^{1-\frac{\alpha}{\beta}}_{L^2(0,1)}$$

for every $\alpha < \beta < 1+\alpha$. We note that, by (4.35), for every $\sigma < \tau, t \leq T$ the first norm is bounded while the second tends to zero as $\tau \to t$ because $u^k \in C([\sigma,T];\mathcal{D}_\alpha)$. Hence, $D^\alpha u^k \cdot \eta \in C((\sigma,T];\mathcal{D}_\alpha)$. Furthermore, $D^\alpha u^k_t \cdot \eta \in C((\sigma,T];L^2(0,1))$. Indeed, $A_2(t)u^k \in C((\sigma,T];L^2(0,1))$ and by (4.34) $A_2(t)u^k \in L^\infty_{loc}((\sigma,T];H^\beta(0,1))$ for every $\beta \in (0,\alpha+1)$. Thus, applying again interpolation estimate, in particular we obtain that $A_2(t)u^k \in C((\sigma,T];H^1(0,1))$ and hence $u^k_t \in C((\sigma,T];H^1(0,1))$ which implies $D^\alpha u^k_t \cdot \eta \in C((\sigma,T];L^2(0,1))$. Applying the Sobolev embedding we obtain that $u^k_t(0,t) \in C((\sigma,T])$ and hence

$$\partial^\alpha u^k_t \cdot \eta = D^\alpha u^k_t \cdot \eta + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \eta \cdot u^k_t(0,t)$$

belongs to $C((\sigma,T];L^2(0,1))$. Finally, we may apply Proposition 2.17 to obtain that $\partial^\alpha u^k \cdot \eta$ satisfies the integral identity

$$(\partial^\alpha u^k \cdot \eta)(x,t) = G(t,\sigma)(\partial^\alpha \varphi^k \cdot \eta)(x) + \int_\sigma^t G(t,\tau) F^k(x,\tau) d\tau, \tag{4.36}$$

where $\{G(t,\tau),\ \sigma \le \tau \le t \le T\}$ denotes an evolution operator generated by the family $A_2(t)$. We fix $\gamma \in (0, 1+\alpha)$, then

$$\left\|(\partial^\alpha u^k \cdot \eta)(\cdot,t)\right\|_{[L^2,\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} \le \left\|G(t,\sigma)(\partial^\alpha \varphi^k \cdot \eta)\right\|_{[L^2,\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} + \int_\sigma^t \left\|G(t,\tau)F^k(\cdot,\tau)\right\|_{[L^2,\mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} d\tau$$

and we apply estimate (4.11) to obtain

$$\left\|(\partial^\alpha u^k \cdot \eta)(\cdot,t)\right\|_{H^\gamma(0,1)} \le c(t-\sigma)^{-\frac{\gamma}{1+\alpha}} \left\|\partial^\alpha \varphi^k \cdot \eta\right\|_{L^2(0,1)} + c\int_\sigma^t (t-\tau)^{-\frac{\gamma}{1+\alpha}} \left\|F^k(\cdot,\tau)\right\|_{L^2(0,1)} d\tau.$$

Using the estimate (4.33) we get that for every $0 < \bar{\gamma} < \frac{1}{2}$ there holds

$$\left\|(\partial^\alpha u^k \cdot \eta)(\cdot,t)\right\|_{H^\gamma(0,1)}$$
$$\le c(t-\sigma)^{-\frac{\gamma}{1+\alpha}} \left\|\partial^\alpha \varphi^k\right\|_{L^2(0,\omega_*)} + c(\varepsilon,\omega)\int_\sigma^t (t-\tau)^{-\frac{\gamma}{1+\alpha}}(\tau-\sigma)^{\frac{\bar{\gamma}}{1+\alpha}-1} d\tau \left\|u_\sigma\right\|_{H^{\bar{\gamma}}(0,1)}.$$

Applying Remark 2.6 together with the estimate (4.32) and (4.25), we obtain that for every $\frac{1}{2} < \gamma < 1+\alpha$ there holds

$$\left\|(D^\alpha u^k \cdot \eta)(\cdot,t)\right\|_{H^\gamma(0,1)} \le c(\varepsilon,\omega)[(t-\sigma)^{-\frac{\gamma}{1+\alpha}} \left\|u_\sigma\right\|_{{}_0H^\alpha(0,\omega_*)} + (t-\sigma)^{\frac{\bar{\gamma}-\gamma}{1+\alpha}} \left\|u_\sigma\right\|_{{}_0H^{\bar{\gamma}}(0,1)}].$$

Since γ is arbitrary number from the interval $(\frac{1}{2}, 1+\alpha)$, the estimate above implies the following: for every $\gamma_1 < \alpha$

$$\left\|(\frac{\partial}{\partial x} D^\alpha u^k \cdot \eta)(\cdot,t)\right\|_{H^{\gamma_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}} \left\|u_\sigma\right\|_{{}_0H^\alpha(0,1)}.$$

Recalling that $\eta \equiv 1$ on $[\varepsilon,\omega]$ we obtain the uniform estimate

$$\left\|(\frac{\partial}{\partial x} D^\alpha u^k)(\cdot,t)\right\|_{H^{\gamma_1}(\varepsilon,\omega)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}} \left\|u_\sigma\right\|_{{}_0H^\alpha(0,1)}.$$

Hence, in view of (4.28) we get that for every $t \in (\sigma, T]$

$$\frac{\partial}{\partial x} D^\alpha u^k \rightharpoonup \frac{\partial}{\partial x} D^\alpha u \quad \text{in} \quad H^{\gamma_1}(\varepsilon,\omega).$$

Furthermore, by weak lower semi-continuity of the norm, we arrive at the estimate: for every $0 < \gamma_1 < \alpha$, for every $t \in (\sigma, T)$ there holds

$$\left\|\frac{\partial}{\partial x} D^\alpha u(\cdot,t)\right\|_{H^{\gamma_1}(\varepsilon,\omega)} \le c(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}} \left\|u_\sigma\right\|_{{}_0H^\alpha(0,1)},$$

where $c = c(\alpha, b, M, \varepsilon, \omega, T, \gamma_1)$. This together with (4.3) finishes the proof. □

Now, we present a more technical result which is necessary to increase the regularity of solutions in the case $0 < \alpha \le \frac{1}{2}$.

Lemma 4.5. *Let $0 < \alpha \le \frac{1}{2}$. Let us assume that $u_\sigma \in H^\beta_{loc}(0,1) \cap H^{\bar{\gamma}}(0,1)$, where $\frac{1}{2} < \beta < 1$ and $0 < \bar{\gamma} < \frac{1}{2}$ are fixed. We denote by u the solution to the equation*

$$\begin{cases} u_t = A_2(t)u & \text{for } 0 < x < 1,\ 0 \le \sigma < t < T, \\ u(x,\sigma) = u_\sigma(x) & \text{for } 0 < x < 1, \end{cases} \tag{4.37}$$

given by the evolution operator generated by the family $A_2(t)$. Then, for every $\max\{\beta - \alpha, \beta - \bar{\gamma}\} < \beta_1 < \beta$, for every $0 < \varepsilon < \omega < 1$, there exists a positive constant $c = c(\alpha, b, M, T, \varepsilon, \omega, \beta, \beta_1)$, such that for every $t \in (\sigma, T]$ there holds

$$\left\|A_2(t)u(\cdot,t)\right\|_{H^{\beta_1}(\varepsilon,\omega)} \le c(t-\sigma)^{-\frac{\beta_1-\beta+\alpha+1}{1+\alpha}} (\left\|u_\sigma\right\|_{H^\beta(\frac{\varepsilon}{2},\frac{1+\omega}{2})} + \left\|u_\sigma\right\|_{H^{\bar{\gamma}}(0,1)}).$$

Proof. We will modify the proof of Lemma 4.4. At first we fix $0<\varepsilon<\omega<1$ and we set $\omega_* = \frac{1+\omega}{2}$. We choose a sequence $\{\varphi^k\} \subseteq \mathcal{D}_\alpha$ such that

$$\varphi^k(0)=0,\ \ \varphi^k \to u_\sigma \text{ in } H^\beta(\varepsilon/2,\omega_*) \ \text{ and } \ \varphi^k \to u_\sigma \text{ in } H^{\bar\gamma}(0,1). \tag{4.38}$$

Let us justify that such sequence exists. Let us take a sequence $\{\Phi^k\} \subseteq C^\infty[\varepsilon/2,\omega_*]$ such that $\Phi^k \to u_\sigma$ in $H^\beta(\varepsilon/2,\omega_*)$, a sequence $\{\Phi_1^k\} \subseteq C_0^\infty[0,\varepsilon/2]$ such that $\Phi_1^k \to u_\sigma$ in $H^{\bar\gamma}(0,\varepsilon/2)$ and a sequence $\{\Phi_2^k\} \subseteq C_0^\infty[\omega_*,1]$ such that $\Phi_2^k \to u_\sigma$ in $H^{\bar\gamma}(\omega_*,1)$. Then let us define

$$\varphi^k(x) = \begin{cases} \Phi_1^k + \rho_1^k & \text{if } x\in[0,\varepsilon/2) \\ \Phi^k & \text{if } x\in[\varepsilon/2,\omega_*] \\ \Phi_2^k+\rho_2^k & \text{if } x\in(\omega_*,1], \end{cases}$$

where the sequences $\{\rho_1^k\}$ and $\{\rho_1^k\}$ are defined as follows. $\{\rho_1^k\}$ is a sequence of smooth functions such that $\rho_1^k \equiv 0$ on $[0,\frac{\varepsilon}{2}-\frac{1}{k}]$, $\rho_1^k(\varepsilon/2)=\Phi^k(\varepsilon/2)$, $\frac{d^-}{dx}\rho_1^k(\varepsilon/2) = \frac{d^+}{dx}\Phi^k(\varepsilon/2)$ and $\frac{d^{2-}}{dx^2}\rho_1^k(\varepsilon/2) = \frac{d^{2+}}{dx^2}\Phi^k(\varepsilon/2)$. Analogously, $\{\rho_2^k\}$ is a sequence of smooth functions such that $\rho_2^k\equiv 0$ on $[w_*+\frac{1}{k},1]$ and $\rho_2^k(\omega_*)=\varphi^k(\omega_*)$, $\frac{d^+}{dx}\rho_2^k(\omega_*) = \frac{d^-}{dx}\Phi^k(\omega_*)$ and $\frac{d^{2+}}{dx^2}\rho_2^k(\omega_*) = \frac{d^{2-}}{dx^2}\Phi^k(\omega_*)$. Since for k large enough there hold $\left|\Phi^k(\varepsilon/2)\right| \le 2\left|u_\sigma(\varepsilon/2)\right|$ and $\left|\Phi^k(\omega_*)\right| \le 2\left|u_\sigma(\omega_*)\right|$, we note that ρ_1^k and ρ_2^k may be chosen in such a way that there exists $c>0$ such that $\left|\frac{d}{dx}\rho_1^k\right| \le ck$ and $\left|\frac{d}{dx}\rho_2^k\right|\le ck$. We note that from the construction $\{\varphi^k\}\subseteq\mathcal{D}_\alpha$. The convergence of φ^k to u_σ in $H^\beta(\varepsilon/2,\omega_*)$ is straightforward, while the convergence in $H^{\bar\gamma}(0,1)$ follows from the fact that $\rho_1^k\to 0$ in $H^{\bar\gamma}(0,\varepsilon/2)$ and $\rho_2^k \to 0$ in $H^{\bar\gamma}(\omega_*,1)$. The two last convergence to zero may be shown as in the proof of Lemma 4.4.

As in the proof of Lemma 4.4, we obtain that the solution to

$$\begin{cases} u_t^k = A_2(t)u^k & \text{for } 0<x<1,\ \ 0\le\sigma<t<T, \\ u^k(x,\sigma)=\varphi^k(x) & \text{for } 0<x<1, \end{cases} \tag{4.39}$$

satisfies for every $0\le\bar\gamma_1<\bar\gamma$

$$\left\|A_2(t)(u-u^k)(\cdot,t)\right\|_{H^{\bar\gamma_1}(0,1)} \le c(t-\sigma)^{-1+\frac{\bar\gamma-\bar\gamma_1}{1+\alpha}}\left\|u_\sigma-\varphi^k\right\|_{H^{\bar\gamma}(0,1)}$$

and for every $t\in(\sigma,T]$

$$\frac{\partial}{\partial x}D^\alpha u^k \to \frac{\partial}{\partial x}D^\alpha u \text{ in } H^{\bar\gamma_1}(0,1). \tag{4.40}$$

Moreover, for k large enough and every $0\le\bar\gamma_1<\bar\gamma$ we get

$$\left\|A_2(t)u^k(\cdot,t)\right\|_{H^{\bar\gamma_1}(0,1)} \le c(t-\sigma)^{-1+\frac{\bar\gamma-\bar\gamma_1}{1+\alpha}}\left\|u_\sigma\right\|_{H^{\bar\gamma}(0,1)}. \tag{4.41}$$

We introduce a smooth nonnegative function $\eta\equiv 0$ on $[0,\varepsilon/2]\cup[\omega_*,1]$, $\eta\equiv 1$ on $[\varepsilon,\omega]$. By regularity of the solutions to approximate problem (4.39) we note that we may apply the operator ∂^β to (4.39). Then we multiply the result by η. Making use of Proposition 2.26 we may calculate as follows

$$\eta\partial^\beta\frac{\partial}{\partial x}D^\alpha u^k = -\frac{\beta}{\Gamma(1-\beta)}\int_0^x (x-p)^{-\beta-1}(\eta(x)-\eta(p))\frac{\partial}{\partial x}D^\alpha u^k(p)dp + \partial^\beta(\frac{\partial}{\partial x}D^\alpha u^k\cdot\eta).$$

Using Proposition 2.30 together with the fact that $(D^{\beta}u^k)(0,t)=0$ we obtain the following sequence of identities

$$\frac{\partial}{\partial x}D^{\alpha}u^k=\partial^{\alpha}\frac{\partial}{\partial x}u^k=\partial^{\alpha}D^{1-\beta}D^{\beta}u^k=D^{1+\alpha-\beta}D^{\beta}u^k=\partial^{1+\alpha-\beta}D^{\beta}u^k.$$

We apply again Proposition 2.26 to get

$$\partial^{1+\alpha-\beta}D^{\beta}u^k\cdot\eta=\frac{\beta-\alpha-1}{\Gamma(\beta-\alpha)}\int_0^x(x-p)^{\beta-\alpha-2}(\eta(x)-\eta(p))D^{\beta}u^k(p)dp+D^{1+\alpha-\beta}(\eta D^{\beta}u^k).$$

Hence,

$$\partial^{\beta}(\frac{\partial}{\partial x}D^{\alpha}u^k\cdot\eta)=\partial^{\beta}\frac{\beta-\alpha-1}{\Gamma(\beta-\alpha)}\int_0^x(x-p)^{\beta-\alpha-2}(\eta(x)-\eta(p))D^{\beta}u^k(p)dp+\partial^{\beta}D^{1+\alpha-\beta}(\eta D^{\beta}u^k).$$

We note that by Definition 2.18, identity (2.15) and Proposition 2.22 we may write

$$\partial^{\beta}D^{1+\alpha-\beta}(\eta D^{\beta}u^k)=\frac{\partial}{\partial x}I^{1-\beta}I^{\beta-\alpha}\frac{\partial}{\partial x}(\eta D^{\beta}u^k)=\frac{\partial}{\partial x}I^{1-\alpha}\frac{\partial}{\partial x}(\eta D^{\beta}u^k)=\partial^{\alpha}\frac{\partial}{\partial x}(\eta D^{\beta}u^k).$$

Applying Remark 2.6 we obtain further

$$\partial^{\beta}D^{1+\alpha-\beta}(\eta D^{\beta}u^k)=\partial^{\alpha}\frac{\partial}{\partial x}(\eta\partial^{\beta}u^k)-\partial^{\alpha}\frac{\partial}{\partial x}(\eta\cdot\frac{x^{-\beta}}{\Gamma(1-\beta)}u^k(0,t))=\partial^{\alpha}\frac{\partial}{\partial x}(\eta\partial^{\beta}u^k)+u^k(0,t)\tilde{\eta},$$

where $\tilde{\eta}:=-\partial^{\alpha}\frac{\partial}{\partial x}(\eta\cdot\frac{x^{-\beta}}{\Gamma(1-\beta)})$ is smooth. Summing up the results, if we apply the operator ∂^{β} to (4.39) and multiply the obtained identity by η we arrive at

$$\begin{cases}(\partial^{\beta}u^k\cdot\eta)_t-A_2(t)(\partial^{\beta}u^k\cdot\eta)=F^k & \text{for } 0<x<1,\ 0\le\sigma<t<T,\\ (\partial^{\beta}u^k\cdot\eta)(\cdot,\sigma)=\partial^{\beta}\varphi^k\cdot\eta & \text{for } 0<x<1,\end{cases}\tag{4.42}$$

where

$$F^k:=-\frac{1}{s^{1+\alpha}(t)}\frac{\beta}{\Gamma(1-\beta)}\int_0^x(x-p)^{-\beta-1}(\eta(x)-\eta(p))\frac{\partial}{\partial x}D^{\alpha}u^k(p)dp$$

$$+\frac{1}{s^{1+\alpha}(t)}\frac{\beta-\alpha-1}{\Gamma(\beta-\alpha)}\partial^{\beta}\int_0^x(x-p)^{\beta-\alpha-2}(\eta(x)-\eta(p))D^{\beta}u^k(p)dp+u^k(0,t)\frac{\tilde{\eta}}{s^{1+\alpha}(t)}=:\sum_{i=1}^{3}F_i^k.$$

We will prove a uniform estimate on the L^2-norm of F^k. For a sake of clarity we neglect in notation the dependance in c of other constants then ω,ε. At first, we note that as in the proof of (4.33), by continuity of fractional integral in L^2 we obtain

$$\left\|F_1^k(\cdot,t)\right\|_{L^2(0,1)}\le c(\varepsilon,\omega)\left\|A_2(t)u^k\right\|_{L^2(0,1)}.$$

To estimate F_2^k we note that

$$s^{1+\alpha}(t)\frac{\Gamma(\beta-\alpha)}{\beta-\alpha-1}\Gamma(1-\beta)F_2^k=\frac{\partial}{\partial x}\int_0^x(x-p)^{-\beta}\int_0^p(p-\tau)^{\beta-\alpha-2}(\eta(p)-\eta(\tau))D^{\beta}u^k(\tau)d\tau dp$$

$$=\frac{\partial}{\partial x}\int_0^x D^{\beta}u^k(\tau)\int_{\tau}^x(x-p)^{-\beta}(p-\tau)^{\beta-\alpha-2}(\eta(p)-\eta(\tau))dpd\tau=\left\{\begin{array}{c}p=\tau+w(x-\tau)\\ dp=(x-\tau)dw\end{array}\right\}$$

$$=\frac{\partial}{\partial x}\int_0^x D^{\beta-\alpha}D^{\alpha}u^k(\tau)(x-\tau)^{-\alpha-1}\int_0^1(1-w)^{-\beta}w^{\beta-\alpha-2}(\eta(\tau+w(x-\tau))-\eta(\tau))dwd\tau$$

$$=\frac{\partial}{\partial x}\int_0^x\int_0^{\tau}\frac{(\tau-p)^{\alpha-\beta}}{\Gamma(1-\beta+\alpha)}\frac{\partial}{\partial x}D^{\alpha}u^k(p)dp(x-\tau)^{-\alpha-1}\int_0^1\frac{(\eta(\tau+w(x-\tau))-\eta(\tau))}{(1-w)^{\beta}w^{2+\alpha-\beta}}dwd\tau,$$

where in the third identity we used the fact that $D^{\alpha}u^k(0,t)=0$ and Proposition 2.30. Applying the Fubini theorem and the substitution $\tau=p+a(x-p)$ we obtain further

$$s^{1+\alpha}(t)\frac{\Gamma(\beta-\alpha)}{\beta-\alpha-1}\Gamma(1-\beta)F_2^k=\frac{1}{\Gamma(1-\beta+\alpha)}\frac{\partial}{\partial x}\int_0^x\frac{\partial}{\partial x}D^{\alpha}u^k(p)(x-p)^{-\beta}H(x,p)dp,$$

where

$$H(x,p)=\int_0^1 a^{\alpha-\beta}(1-a)^{-\alpha-1}\int_0^1 \frac{\eta(p+a(x-p)+w(x-p)(1-a))-\eta(p+a(x-p))}{(1-w)^{\beta}w^{2+\alpha-\beta}}dwda.$$

We note that for every $a,w,p,x\in(0,1)$, $p\neq x$ we have

$$\left|\frac{(\eta(p+a(x-p)+w(x-p)(1-a))-\eta(p+a(x-p)))}{(x-p)(1-a)w}\right|\leq \|\eta\|_{W^{1,\infty}(0,1)}. \tag{4.43}$$

Hence, denoting by $B(\cdot,\cdot)$ the Beta function we may estimate as follows

$$\begin{aligned}\left|(x-p)^{-\beta}H(x,p)\right| &\leq \|\eta\|_{W^{1,\infty}(0,1)}|x-p|^{1-\beta}\int_0^1 a^{\alpha-\beta}(1-a)^{-\alpha}da\int_0^1(1-w)^{-\beta}w^{\beta-\alpha-1}dw \\ &= |x-p|^{1-\beta}B(1-\alpha,\alpha-\beta+1)B(1-\beta,\beta-\alpha)\to 0 \text{ as } p\to x.\end{aligned} \tag{4.44}$$

Having in mind that $\frac{\partial}{\partial x}D^{\alpha}u^k$ is bounded with respect to space for any $t\in(\sigma,T)$ and applying estimate (4.43), it is not difficult to pass with differentiation under the integral sign by virtue of the Lebesgue dominated convergence theorem. Thus, performing differentiation we arrive at the following identity

$$\begin{aligned}&s^{1+\alpha}(t)\frac{\Gamma(\beta-\alpha)}{\beta-\alpha-1}\Gamma(1-\beta)\Gamma(1-\beta+\alpha)F_2^k \\ &=\int_0^x \frac{\partial}{\partial x}D^{\alpha}u^k(p)(x-p)^{-\beta}\frac{\partial}{\partial x}H(x,p)dp-\beta\int_0^x\frac{\partial}{\partial x}D^{\alpha}u^k(p)(x-p)^{-\beta}\frac{H(x,p)}{x-p}dp.\end{aligned}$$

We note that

$$\begin{aligned}\frac{\partial}{\partial x}H(x,p) &= \int_0^1 a^{\alpha-\beta}(1-a)^{-\alpha}\int_0^1(1-w)^{-\beta}w^{\beta-\alpha-1}\eta'(p+a(x-p)+w(x-p)(1-a))dwda \\ &+\int_0^1 a^{1+\alpha-\beta}(1-a)^{-\alpha-1}\int_0^1\frac{\eta'(p+a(x-p)+w(x-p)(1-a))-\eta'(p+a(x-p))}{(1-w)^{\beta}w^{2+\alpha-\beta}}dwda.\end{aligned}$$

Hence, applying (4.43) to η' we obtain

$$\left|\frac{\partial}{\partial x}H(x,p)\right|\leq c\|\eta\|_{W^{2,\infty}(0,1)}.$$

This together with estimate (4.44) leads to

$$\left|\frac{\partial}{\partial x}H(x,p)\right|+\left|\frac{H(x,p)}{x-p}\right|\leq c(\varepsilon,\omega) \text{ for every } 0\leq p<x\leq 1.$$

Finally,

$$\left\|F_2^k(\cdot,t)\right\|_{L^2(0,1)}\leq c(\varepsilon,\omega)\left\|I^{1-\beta}\left|A_2(t)u^k\right|\right\|_{L^2(0,1)}\leq c(\varepsilon,\omega)\left\|A_2(t)u^k(\cdot,t)\right\|_{L^2(0,1)}.$$

To estimate F_3^k it is enough to apply the Sobolev embedding

$$\left\|\tilde{\eta}u^k(0,t)\right\|_{L^2(0,1)}\leq c(\varepsilon,\omega)\left\|u^k(\cdot,t)\right\|_{C([0,1])}\leq c(\varepsilon,\omega)\left\|A_2(t)u^k(\cdot,t)\right\|_{L^2(0,1)}.$$

Moreover, similarly as in estimate (4.32) for every $\frac{1}{2}<\delta<1+\alpha$ we have

$$\left|u^k(0,t)\right|\leq c(t-\sigma)^{\frac{\bar{\gamma}-\delta}{1+\alpha}}\left\|\varphi^k\right\|_{H^{\bar{\gamma}}(0,1)}. \tag{4.45}$$

Summing up the estimates for F_i^k, $i=1,2,3$ and applying (4.41) we obtain that

$$\left\|F^k(\cdot,t)\right\|_{L^2(0,1)}\leq c(\varepsilon,\omega)(t-\sigma)^{\frac{\bar{\gamma}}{1+\alpha}-1}\|u_{\sigma}\|_{H^{\bar{\gamma}}(0,1)}. \tag{4.46}$$

We will show that F^k and $\partial^\beta u^k \cdot \eta$ satisfy the assumptions of Proposition 2.17. Since $u^k \in C([\sigma,T];\mathcal{D}_\alpha)$ by continuity of s and estimate above we obtain that $F^k \in L^1(\sigma,T;L^2(0,1)) \cap C((\sigma,T];L^2(0,1))$. Furthermore, $\partial^\beta u^k \cdot \eta \in C([\sigma,T];L^2(0,1))$. Let us check that $\partial^\beta u^k \cdot \eta \in C((\sigma,T];\mathcal{D}_\alpha)$. Again, since $\partial^\beta u^k \cdot \eta = D^\beta u^k \cdot \eta + \frac{x^{-\alpha}}{\Gamma(1-\alpha)} u^k(0,t) \cdot \eta$, it is enough to check the regularity of $D^\beta u^k \cdot \eta$.

From $\varphi^k \in \mathcal{D}_\alpha$ and estimate (4.12) we infer that

$$A_2(t)u^k \in L^\infty_{loc}((\sigma,T];H^\gamma(0,1)) \text{ for every } 0<\gamma<1+\alpha. \tag{4.47}$$

At first, we will show the following estimate

$$\left\|\eta D^\beta u^k(\cdot,t)\right\|_{\mathcal{D}_\alpha} \le c(\varepsilon,\omega) \left\|\frac{\partial}{\partial x} D^\alpha u^k(\cdot,t)\right\|_{H^\beta(0,1)} \quad \text{for every} \quad t \in (\sigma,T]. \tag{4.48}$$

We note that in the following calculations we may replace $u^k(\cdot,t)$ with the difference $u^k(\cdot,t) - u^k(\cdot,\tau)$ for any $\tau, t \in (\sigma,T]$. Applying Poincaré inequality we have

$$\left\|\eta D^\beta u^k\right\|_{\mathcal{D}_\alpha} \le \left\|\frac{\partial}{\partial x}(\eta D^\beta u^k)\right\|_{{}_0H^\alpha(0,1)} = \left\|\eta' D^\beta u^k\right\|_{{}_0H^\alpha(0,1)} + \left\|\eta \frac{\partial}{\partial x} D^\beta u^k\right\|_{{}_0H^\alpha(0,1)}. \tag{4.49}$$

With the first term we deal as follows. We apply Proposition 2.32 together with Proposition 2.30 and the fact that $D^\alpha u^k(0,t) = 0$ to get

$$\left\|\eta' D^\beta u^k\right\|_{{}_0H^\alpha(0,1)} \le c\left\|\partial^\alpha(\eta' D^{\beta-\alpha} D^\alpha u^k)\right\|_{L^2(0,1)}$$

$$\le \left\|\eta' \partial^\alpha D^{\beta-\alpha} D^\alpha u^k\right\|_{L^2(0,1)} + \frac{\alpha}{\Gamma(1-\alpha)} \left\|\int_0^x (x-p)^{-\alpha-1}(\eta'(x)-\eta'(p)) D^\beta u^k(p)dp\right\|_{L^2(0,1)}.$$

We note that in the last inequality we applied Proposition 2.26. By the continuity of fractional integral in L^2 we obtain that

$$\left\|\int_0^x (x-p)^{-\alpha-1}(\eta'(x)-\eta'(p)) D^\beta u^k(p)dp\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|D^\beta u^k\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|u^k_x\right\|_{L^2(0,1)}.$$

Furthermore, from Proposition 2.30 we infer that

$$\left\|\eta' \partial^\alpha D^{\beta-\alpha} D^\alpha u^k\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|D^\beta D^\alpha u^k\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|\frac{\partial}{\partial x} D^\alpha u^k\right\|_{L^2(0,1)}.$$

Hence, we arrive at

$$\left\|\eta' D^\beta u^k\right\|_{{}_0H^\alpha(0,1)} \le c(\varepsilon,\omega)\left\|\frac{\partial}{\partial x} D^\alpha u^k\right\|_{L^2(0,1)}. \tag{4.50}$$

Let us now focus on the last term in (4.49). Applying Proposition 2.30, the fact that $D^\alpha u^k(0,t) = 0$ and Remark 2.7 we obtain the following sequence of identities

$$\frac{\partial}{\partial x} D^\beta u^k = \frac{\partial}{\partial x} D^{\beta-\alpha} D^\alpha u^k = \partial^{\beta-\alpha}\partial^{1-(\beta-\alpha)} D^{\beta-\alpha} D^\alpha u^k = \partial^{\beta-\alpha}\frac{\partial}{\partial x} D^\alpha u^k.$$

Hence, by Proposition 2.26

$$\eta\frac{\partial}{\partial x} D^\beta u^k = \frac{\alpha-\beta}{\Gamma(1-(\alpha-\beta))}\int_0^x (x-p)^{\alpha-\beta}\frac{\eta(x)-\eta(p)}{x-p}\frac{\partial}{\partial x} D^\alpha u^k(p)dp + \partial^{\beta-\alpha}(\eta\frac{\partial}{\partial x} D^\alpha u^k). \tag{4.51}$$

In order to estimate the H^α-norm of the second term on the r.h.s we apply Proposition 2.32 and Proposition 2.30 in the following way

$$\left\|\partial^{\beta-\alpha}(\eta\frac{\partial}{\partial x} D^\alpha u^k)\right\|_{{}_0H^\alpha(0,1)} \le c\left\|\partial^\alpha D^{\beta-\alpha}(\eta\frac{\partial}{\partial x} D^\alpha u^k)\right\|_{L^2(0,1)}$$

$$= c\left\|D^\beta(\eta\frac{\partial}{\partial x}D^\alpha u^k)\right\|_{L^2(0,1)} \le c\left\|\eta\frac{\partial}{\partial x}D^\alpha u^k\right\|_{{}_0H^\beta(0,1)}. \tag{4.52}$$

Let us denote

$$J^k \equiv \int_0^x (x-p)^{\alpha-\beta}\frac{\eta(x)-\eta(p)}{x-p}\frac{\partial}{\partial x}D^\alpha u^k(p)dp.$$

We will estimate the norm of J^k in the space ${}_0H^\alpha(0,1)$. In view of Proposition 2.32, it is enough to estimate the L^2- norm of $\partial^\alpha J^k$. We may calculate as follows

$$\Gamma(1-\alpha)\partial^\alpha J^k = \frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}\int_0^p (p-\tau)^{\alpha-\beta-1}(\eta(p)-\eta(\tau))\frac{\partial}{\partial x}D^\alpha u^k(\tau)d\tau dp$$

$$= \frac{\partial}{\partial x}\int_0^x \frac{\partial}{\partial x}D^\alpha u^k(\tau)\int_\tau^x (x-p)^{-\alpha}(p-\tau)^{\alpha-\beta-1}(\eta(p)-\eta(\tau))dpd\tau = \left\{\begin{array}{c} p=\tau+w(x-\tau) \\ dp=(x-\tau)dw \end{array}\right\}$$

$$= \frac{\partial}{\partial x}\int_0^x \frac{\partial}{\partial x}D^\alpha u^k(\tau)(x-\tau)^{-\beta}\int_0^1 (1-w)^{-\alpha}w^{\alpha-\beta-1}(\eta(\tau+w(x-\tau))-\eta(\tau))dwd\tau.$$

We note that we may differentiate under the integral sign. Indeed,

$$|\eta(\tau+w(x-\tau))-\eta(\tau)| \le \|\eta\|_{W^{1,\infty}(0,1)}w(x-\tau) \text{ for every } w\in(0,1),\ \ 0\le\tau<x\le 1.$$

Hence,

$$\left|\frac{\partial}{\partial x}D^\alpha u^k(\tau)(x-\tau)^{-\beta}\int_0^1 (1-w)^{-\alpha}w^{\alpha-\beta-1}(\eta(\tau+w(x-\tau))-\eta(\tau))dw\right|$$

$$\le \|\eta\|_{W^{1,\infty}(0,1)}B(1-\alpha,1+\alpha-\beta)\left|\frac{\partial}{\partial x}D^\alpha u^k(\tau)(x-\tau)^{1-\beta}\right| \to 0 \text{ as } \tau\to x^-.$$

Recalling that $\frac{\partial}{\partial x}D^\alpha u^k$ is bounded with respect to space for any positive time, we may apply the Lebesgue dominated convergence theorem to obtain that

$$\Gamma(1-\alpha)\partial^\alpha J^k = \int_0^x \frac{\partial}{\partial x}D^\alpha u^k(\tau)(x-\tau)^{-\beta}\int_0^1 (1-w)^{-\alpha}w^{\alpha-\beta}\eta'(\tau+w(x-\tau))dwd\tau$$

$$-\beta\int_0^x \frac{\partial}{\partial x}D^\alpha u^k(\tau)(x-\tau)^{-\beta-1}\int_0^1 (1-w)^{-\alpha}w^{\alpha-\beta-1}(\eta(\tau+w(x-\tau))-\eta(\tau))dw.$$

Estimating the L^2- norm of expression above we arrive at

$$\left\|J^k\right\|_{{}_0H^\alpha(0,1)} \le c\|\eta\|_{W^{1,\infty}(0,1)}\left\|I^{1-\beta}\left|\frac{\partial}{\partial x}D^\alpha u^k\right|\right\|_{L^2(0,1)} \le c(\varepsilon,\omega)\left\|\frac{\partial}{\partial x}D^\alpha u^k\right\|_{L^2(0,1)}, \tag{4.53}$$

where we applied boundedness of fractional integral in L^2. Combining the last estimate together with (4.50) and (4.52) we obtain (4.48). Applying (4.48) on a difference $u^k(\cdot,t)-u^k(\cdot,\tau)$ together with interpolation estimate [19, Corollary 1.2.7.] we obtain that for every $\beta<\gamma<1+\alpha$ there holds

$$\left\|\eta D^\beta(u^k(\cdot,t)-u^k(\cdot,\tau))\right\|_{\mathcal{D}_\alpha} \le c(\varepsilon,\omega)\left\|\frac{\partial}{\partial x}D^\alpha(u^k(\cdot,t)-u^k(\cdot,\tau))\right\|_{H^\beta(0,1)}$$

$$\le c(\varepsilon,\omega)\left\|\frac{\partial}{\partial x}D^\alpha(u^k(\cdot,t)-u^k(\cdot,\tau))\right\|_{H^\gamma(0,1)}^{\frac{\beta}{\gamma}}\left\|\frac{\partial}{\partial x}D^\alpha(u^k(\cdot,t)-u^k(\cdot,\tau))\right\|_{L^2(0,1)}^{1-\frac{\beta}{\gamma}}.$$

The second term tends to zero as $\tau \to t$ while the first one is bounded for $\sigma < \tau, t \leq T$ due to (4.47). Hence, we have shown that $D^\beta u^k \in C((\sigma, T]; \mathcal{D}_\alpha)$ and thus $\partial^\beta u^k \cdot \eta \in C((\sigma, T]; \mathcal{D}_\alpha)$. It remains to prove $\partial^\beta u_t^k \cdot \eta \in C((\sigma, T]; L^2(0,1))$. From Proposition 2.26 we infer that

$$\partial^\beta u_t^k \cdot \eta = \partial^\beta (u_t^k \cdot \eta) - \frac{\beta}{1-\beta} \int_0^x (x-p)^{-\beta-1} (\eta(x) - \eta(p)) u_t^k(p) dp$$

We note that for every $\tau, t \in (\sigma, T)$

$$\left\| \int_0^x (x-p)^{-\beta-1} (\eta(x) - \eta(p)) [u_t^k(p,t) - u_t^k(p,\tau)] dp \right\|_{L^2(0,1)} \leq c(\varepsilon, \omega) \left\| u_t^k(\cdot, t) - u_t^k(\cdot, \tau) \right\|_{L^2(0,1)}$$

and the last expression tends to zero as $\tau \to t$ because $u_t^k \in C((\sigma, T]; L^2(0,1))$. Applying Proposition 2.32 we obtain for every $\tau, t \in (\sigma, T)$

$$\left\| \partial^\beta (u_t^k \cdot \eta)(\cdot, t) - \partial^\beta (u_t^k \cdot \eta)(\cdot, \tau) \right\|_{L^2(0,1)} \leq c \left\| \eta (u_t^k(\cdot, t) - u_t^k(\cdot, \tau)) \right\|_{{}_0H^\beta(0,1)}.$$

The last term tends to zero as $\tau \to t$ for τ, t positive. Indeed, since $A_2(t) u^k = u_t^k$, it is enough to apply the interpolation estimate together with (4.47) and the fact that $A_2(t) u^k \in C([\sigma, T]; L^2(0,1))$. Hence, we have shown that $\partial^\beta u_t^k \cdot \eta \in C((\sigma, T]; L^2(0,1))$. Finally, we are able to apply Proposition 2.17 to deduce that $\partial^\beta u^k \cdot \eta$ satisfies the integral identity

$$(\partial^\beta u^k \cdot \eta)(x,t) = G(t,\sigma)(\partial^\beta \varphi^k \cdot \eta) + \int_\sigma^t G(t,\tau) F(x,\tau) d\tau, \tag{4.54}$$

where $\{G(t,\tau),\ \sigma \leq \tau \leq t \leq T\}$ denotes an evolution operator generated by the family $A_2(t)$. We fix $\gamma \in (0, 1+\alpha)$, then we may write

$$\left\| (\partial^\beta u^k \cdot \eta)(\cdot, t) \right\|_{[L^2, \mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} \leq \left\| G(t,\sigma)(\partial^\beta \varphi^k \cdot \eta) \right\|_{[L^2, \mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} + \int_\sigma^t \| G(t,\tau) F(\cdot, \tau) \|_{[L^2, \mathcal{D}_\alpha]_{\frac{\gamma}{1+\alpha}}} d\tau.$$

Hence, by estimate (4.11) we have

$$\left\| (\partial^\beta u^k \cdot \eta)(\cdot, t) \right\|_{H^\gamma(0,1)} \leq c(t-\sigma)^{-\frac{\gamma}{1+\alpha}} \left\| \partial^\beta \varphi^k \cdot \eta \right\|_{L^2(0,1)} + c \int_\sigma^t (t-\tau)^{-\frac{\gamma}{1+\alpha}} \left\| F^k(\cdot, \tau) \right\|_{L^2(0,1)} d\tau.$$

Applying (4.46) we obtain that

$$\begin{aligned} &\left\| (\partial^\beta u^k \cdot \eta)(\cdot, t) \right\|_{H^\gamma(0,1)} \\ &\leq c(t-\sigma)^{-\frac{\gamma}{1+\alpha}} \left\| \partial^\beta \varphi^k \cdot \eta \right\|_{L^2(0,1)} + c(\varepsilon, \omega) \int_\sigma^t (t-\tau)^{-\frac{\gamma}{1+\alpha}} (\tau - \sigma)^{\frac{\bar{\gamma}}{1+\alpha} - 1} d\tau \, \| u_\sigma \|_{H^{\bar{\gamma}}(0,1)}. \end{aligned}$$

We note that by Proposition (2.26) we have

$$\partial^\beta \varphi^k \cdot \eta = -\frac{\beta}{\Gamma(1-\beta)} \int_0^x (x-p)^{-\beta} \frac{\eta(x) - \eta(p)}{x-p} \varphi^k(p) dp + \partial^\beta (\varphi^k \cdot \eta).$$

Hence,

$$\left\| \partial^\beta \varphi^k \cdot \eta \right\|_{L^2(0,1)} \leq c(\varepsilon, \omega) \left\| \varphi^k \right\|_{L^2(0,1)} + \left\| \varphi^k \eta \right\|_{{}_0H^\beta(0,1)}$$

and by (4.38)

$$\left\| \partial^\beta \varphi^k \cdot \eta \right\|_{L^2(0,1)} \leq c(\varepsilon, \omega) (\| u_\sigma \|_{H^\beta(\frac{\varepsilon}{2}, \omega_*)} + \| u_\sigma \|_{L^2(0,1)}).$$

Thus, we get that

$$\left\| (\partial^\beta u^k \cdot \eta)(\cdot, t) \right\|_{H^\gamma(0,1)} \leq c(\varepsilon, \omega) [(t-\sigma)^{-\frac{\gamma}{1+\alpha}} (\| u_\sigma \|_{H^\beta(\frac{\varepsilon}{2}, \omega_*)} + \| u_\sigma \|_{L^2(0,1)}) + (t-\sigma)^{\frac{\bar{\gamma}-\gamma}{1+\alpha}} \| u_\sigma \|_{H^{\bar{\gamma}}(0,1)}].$$

Since γ is an arbitrary number from the interval $(0,1+\alpha)$, the estimate above implies the following: for every $\gamma_1 < \alpha$

$$\left\| \frac{\partial}{\partial x}(\partial^\beta u^k \cdot \eta)(\cdot,t) \right\|_{H^{\gamma_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)}).$$

Applying Remark 2.6 and the estimate (4.45) we obtain

$$\left\| \frac{\partial}{\partial x}(D^\beta u^k \cdot \eta(\cdot,t)) \right\|_{H^{\gamma_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)})$$

and in view of (4.41) and (4.50) we have

$$\left\| \frac{\partial}{\partial x}D^\beta u^k \cdot \eta(\cdot,t) \right\|_{H^{\gamma_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)}). \tag{4.55}$$

We will show that this leads to

$$\left\| \eta\frac{\partial}{\partial x}D^\alpha u^k(\cdot,t) \right\|_{H^{\beta_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\beta_1-\beta+\alpha+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)}) \tag{4.56}$$

for every $\max\{\beta-\alpha,\beta-\bar\gamma\} < \beta_1 < \beta$ and where $c = c(\alpha,b,M,T,\varepsilon,\omega,\beta_1,\beta)$. Indeed, making use of estimates (4.41), (4.53) and (4.55) in identity (4.51) we obtain that for any $\max\{0,\alpha-\bar\gamma\} < \gamma_1 < \alpha$ there holds

$$\left\| \partial^{\beta-\alpha}(\eta\frac{\partial}{\partial x}D^\alpha u^k) \right\|_{H^{\gamma_1}(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)}).$$

Hence, for any $\max\{0,\alpha-\bar\gamma\} < \gamma_1 < \alpha$ we have

$$\left\| \partial^{\gamma_1}D^{\beta-\alpha}(\eta\frac{\partial}{\partial x}D^\alpha u^k) \right\|_{L^2(0,1)} \le c(\varepsilon,\omega)(t-\sigma)^{-\frac{\gamma_1+1}{1+\alpha}}(\|u_\sigma\|_{H^\beta(\frac{\varepsilon}{2},\omega_*)} + \|u_\sigma\|_{H^{\bar\gamma}(0,1)}).$$

Taking $\gamma_1 = \beta_1 - \beta + \alpha$, where $\max\{\beta-\bar\gamma,\beta-\alpha\} < \beta_1 < \beta$ and applying Proposition 2.30 and Proposition 2.32 we arrive at (4.56). Estimate (4.56), together with the weak lower semi-continuity of the norm finishes the proof. □

Finally, we are able to improve the space regularity of solutions to (4.7). We decompose interval $(0,1)$ as follows

$$(0,1) = \left(\bigcup_{k=1}^{\infty}\left(\frac{1}{k+1},\frac{1}{k}\right]\right) \setminus \{1\}.$$

Then, for each $\alpha \in (0,1)$ we may choose $k \in \mathbb{N}\setminus\{0\}$ such that $\alpha \in (\frac{1}{k+1},\frac{1}{k}]$. We will discuss separately the case for each $k \in \mathbb{N}\setminus\{0\}$. The proof for $\alpha \in (\frac{1}{2},1)$ requires just one step, however for $\alpha \in (\frac{1}{k+1},\frac{1}{k}]$ we need to repeat the reasoning k times.

Lemma 4.6. *Let us assume that $v_0 \in \mathcal{D}_\alpha$, $\alpha \in (0,1)$. We choose $k \in \mathbb{N}\setminus\{0\}$ such that $\alpha \in (\frac{1}{k+1},\frac{1}{k}]$. Then, for every $\gamma_k \in (\alpha,(k+1)\alpha)$ the solution to (4.7) obtained in Theorem 4.2 satisfies*

$$v \in L^\infty_{loc}(0,T;H^{\gamma_k+1}_{loc}(0,1)) \text{ and } \partial^\alpha v_x \in L^\infty_{loc}(0,T;H^{\gamma_k-\alpha}_{loc}(0,1)). \tag{4.57}$$

Furthermore,

$$v \in L^{\frac{1+\alpha}{k\alpha}}(0,T;H^{\gamma_k+1}_{loc}(0,1)) \text{ and } \partial^\alpha v_x \in L^{\frac{1+\alpha}{k\alpha}}(0,T;H^{\gamma_k-\alpha}_{loc}(0,1)).$$

Proof. Let us denote $\delta = \frac{\alpha}{\alpha+1}$ in the case $\alpha \neq \frac{1}{2}$ and for $\alpha = \frac{1}{2}$ by δ we mean any number from the interval $(0, \frac{1}{3})$. We apply the operator $A_2(t)$ to (4.21), where f is defined in (4.18), and estimate its norm in an interpolation space. Firstly, we consider the case $\alpha \in (\frac{1}{2}, 1)$. In this case, from Theorem 4.2 we deduce that $f \in L^\infty(0,T; {}_0H^{(1+\alpha)\delta}(0,1))$. Thus, by Lemma 4.4 we obtain for any $0 < \varepsilon < \omega < 1$ and any $0 < \theta < \delta$

$$\left\| A_2(t) \int_0^t G(t,\sigma) f(\cdot,\sigma) d\sigma \right\|_{H^{(1+\alpha)\theta}(\varepsilon,\omega)} \leq \int_0^t \| A_2(t) G(t,\sigma) f(\cdot,\sigma) \|_{H^{(1+\alpha)\theta}(\varepsilon,\omega)} d\sigma$$

$$\leq \int_0^t \frac{c}{(t-\sigma)^{1+\theta-\delta}} \| f(\cdot,\sigma) \|_{{}_0H^{(1+\alpha)\delta}(0,1)} d\sigma \leq \frac{cT^{\delta-\theta}}{\delta-\theta} \| f \|_{L^\infty(0,T;{}_0H^\alpha(0,1))},$$

where the constant $c > 0$ comes from Lemma 4.4. By the regularity of the initial condition and estimate (4.12) we have

$$\| A_2(t) G(t,0) v_0 \|_{H^{(1+\alpha)\theta}(0,1)} \leq \frac{c}{t^\theta} \| v_0 \|_{\mathcal{D}_\alpha}.$$

Thus, in view of formula (4.14), we get that for every $0 < \gamma < \alpha$

$$A_2(t) v \in L^\infty_{loc}(0,T; H^\gamma_{loc}(0,1)), \;\; A_2(t) v \in L^{\frac{\alpha+1}{\alpha}}(0,T; H^\gamma_{loc}(0,1)),$$

taking into account (4.3), this leads to

$$\partial^\alpha v_x \in L^\infty_{loc}(0,T; H^\gamma_{loc}(0,1)), \;\; \partial^\alpha v_x \in L^{\frac{\alpha+1}{\alpha}}(0,T; H^\gamma_{loc}(0,1)).$$

Applying Lemma 2.34 we obtain that

$$v_x \in L^\infty_{loc}(0,T; H^{\gamma+\alpha}_{loc}(0,1)) \text{ and } v_x \in L^{\frac{\alpha+1}{\alpha}}(0,T; H^{\gamma+\alpha}_{loc}(0,1)),$$

which finishes the proof in the case $\alpha \in (\frac{1}{2}, 1)$.
In the case $\alpha \leq \frac{1}{2}$, by Theorem 4.2, we have $f \in L^\infty(0,T; [L^2, \mathcal{D}_\alpha]_\delta)$. Thus, by (4.12), we obtain that for any $\theta < \delta$

$$\left\| A_2(t) \int_0^t G(t,\sigma) f(\cdot,\sigma) d\sigma \right\|_{[L^2(0,1),\mathcal{D}_\alpha]_\theta} \leq \int_0^t \| A_2(t) G(t,\sigma) f(\cdot,\sigma) \|_{[L^2(0,1),\mathcal{D}_\alpha]_\theta} d\sigma$$

$$\leq \int_0^t \frac{c}{(t-\sigma)^{1+\theta-\delta}} \| f(\cdot,\sigma) \|_{[L^2(0,1),\mathcal{D}_\alpha]_\delta} d\sigma \leq \frac{cT^{\delta-\theta}}{\delta-\theta} \| f \|_{L^\infty(0,T;{}_0H^\alpha(0,1))}.$$

This together with the estimate

$$\| A_2(t) G(t,0) v_0 \|_{[L^2(0,1),D^\alpha]_\theta} \leq \frac{c}{t^\theta} \| v_0 \|_{\mathcal{D}_\alpha}$$

and formula (4.21), implies that for every $0 < \theta < \delta$

$$A_2(t) v \in L^\infty_{loc}(0,T; [L^2(0,1), \mathcal{D}_\alpha]_\theta) = L^\infty_{loc}(0,T; H^{(1+\alpha)\theta}(0,1)).$$

Hence, in view of (4.3)

$$\partial^\alpha v_x \in L^\infty_{loc}(0,T; H^{\gamma_0}(0,1)) \text{ for every } \gamma_0 \in (0,\alpha).$$

Applying Corollary 2.33 we obtain that

$$\| v_x(\cdot,t) \|_{{}_0H^{\gamma_1}(0,1)} \leq ct^{-\frac{\alpha}{\alpha+1}} \| v_0 \|_{\mathcal{D}_\alpha} \text{ for every } \gamma_1 < 2\alpha. \tag{4.58}$$

Let us denote $\delta_1 = \frac{\gamma_1}{1+\alpha}$. We will discuss firstly the case $\alpha \in (\frac{1}{4}, \frac{1}{2}]$. We note that by (4.58)

$$\| f(\cdot,t) \|_{H^{(1+\alpha)\delta_1}(0,1)} \leq ct^{-\frac{\alpha}{\alpha+1}} \| v_0 \|_{\mathcal{D}_\alpha}.$$

Applying Lemma 4.5, we obtain for every $0<\varepsilon<\omega<1$ and every $\theta<\delta_1$

$$\begin{aligned}\|A_2(t)v(\cdot,t)\|_{H^{(1+\alpha)\theta}(\varepsilon,\omega)} &\le \frac{c}{t^\theta}\|v_0\|_{\mathcal{D}_\alpha}+\int_0^t \|A_2(t)G(t,\sigma)f(\cdot,\sigma)\|_{H^{(1+\alpha)\theta}(\varepsilon,\omega)}\,d\sigma\\ &\le \frac{c}{t^\theta}\|v_0\|_{\mathcal{D}_\alpha}+c\int_0^t \sigma^{-\frac{\alpha}{\alpha+1}}(t-\sigma)^{-1-\theta+\delta_1}d\sigma\,\|v_0\|_{\mathcal{D}_\alpha}\end{aligned}$$

and we arrive at

$$A_2(t)v\in L^\infty_{loc}(0,T;H^{(1+\alpha)\theta}_{loc}(0,1)),\;\; A_2(t)v\in L^{\frac{\alpha+1}{2\alpha}}(0,T;H^{(1+\alpha)\theta}_{loc}(0,1)) \text{ for every } \theta<\delta_1.$$

Thus,

$$\partial^\alpha v_x\in L^\infty_{loc}(0,T;H^{\gamma_1}_{loc}(0,1)),\;\; \partial^\alpha v_x\in L^{\frac{\alpha+1}{2\alpha}}(0,T;H^{\gamma_1}_{loc}(0,1)) \text{ for every } \gamma_1<2\alpha.$$

Applying Lemma 2.34 we get that

$$v_x\in L^\infty_{loc}(0,T;H^{\gamma_2}_{loc}(0,1)),\;\; v_x\in L^{\frac{\alpha+1}{2\alpha}}(0,T;H^{\gamma_2}_{loc}(0,1)) \text{ for every } \gamma_2<3\alpha.$$

In this way we proved the lemma for $\alpha\in(\frac{1}{3},1)$. Let us suppose that $\alpha\in(\frac14,\frac13]$, since $f\in L^{\frac{\alpha+1}{2\alpha}}(0,T;H^{\gamma_2}_{loc}(0,1)\cap H^{\bar\gamma}(0,1))$ for every $0<\bar\gamma<\frac12$, we apply Lemma 4.5 with $\beta=\gamma_2$ together with Lemma 2.34 and we obtain that

$$\partial^\alpha v_x\in L^\infty_{loc}(0,T;H^{\gamma_2}_{loc}(0,1)) \text{ and } v_x\in L^\infty_{loc}(0,T;H^{\gamma_3}_{loc}(0,1)) \text{ for every } \gamma_3<4\alpha.$$

In general case, we proceed as follows. For $\alpha\in(\frac{1}{k+1},\frac1k]$, $k\ge 2$ we apply to (4.21) the estimate (4.12) with $\delta_n=\frac{\gamma_n}{\alpha+1}$, $\gamma_n<(n+1)\alpha$ for $n=0,\dots,\lceil\frac{k-1}{2}\rceil-1$. In this way we obtain that

$$v_x\in L^\infty_{loc}(0,T;{}_0H^{\gamma_{\lceil(k-1)/2\rceil}}(0,1)) \tag{4.59}$$

and

$$\|f(\cdot,t)\|_{H^{\gamma_{\lceil(k-1)/2\rceil}}(0,1)}\le ct^{-\frac{\gamma_{\lceil(k-1)/2\rceil}}{\alpha+1}}\|v_0\|_{\mathcal{D}_\alpha}.$$

Then we apply to (4.21) Lemma 4.5 together with Lemma 2.34 $\lceil\frac{k}{2}\rceil$ times with $\beta=\gamma_n$ for $n=\lceil\frac{k-1}{2}\rceil,\dots,k-1$ to obtain

$$\partial^\alpha v_x\in L^\infty_{loc}(0,T;H^{\gamma_{k-1}}_{loc}(0,1)),\;\; v_x\in L^\infty_{loc}(0,T;H^{\gamma_k}_{loc}(0,1)),\;\; v_x\in L^{\frac{\alpha+1}{k\alpha}}(0,T;H^{\gamma_k}_{loc}(0,1)),$$

which finishes the proof. □

In Theorem 4.2 we have obtained the solution to (4.7) belonging to $C([0,T];\mathcal{D}_\alpha)$. By Lemma 4.6 we may deduce local continuity of the solution with values in more regular spaces. We establish this result in the following corollary.

Corollary 4.7. *Let us assume that $v_0\in\mathcal{D}_\alpha$. Let v be a solution to (4.7) given by Theorem 4.2. Let $\alpha\in(0,1)$, we choose $k\in\mathbb{N}\setminus\{0\}$ such that $\alpha\in(\frac{1}{k+1},\frac1k]$. Then, for every $\alpha<\gamma_k<(k+1)\alpha$ there holds*

$$v\in C((0,T];H^{\gamma_k+1}_{loc}(0,1)) \text{ and } \partial^\alpha v_x\in C((0,T];H^{\gamma_k-\alpha}_{loc}(0,1)). \tag{4.60}$$

Furthermore,

$$v_x\in C((0,T];{}_0C[0,1]) \text{ for } \alpha\in(0,\frac12] \text{ and } v_x\in C([0,T];{}_0C[0,1]) \text{ for } \alpha\in(\frac12,1). \tag{4.61}$$

Proof. Theorem 4.2 states that $v \in C([0,T];\mathcal{D}_\alpha)$. Since for arbitrary $0<\varepsilon<\omega<1$ and for every $\alpha < \overline{\gamma_k} < \gamma_k < (k+1)\alpha$ there holds

$$H^{\overline{\gamma_k}+1}(\varepsilon,\omega) = [H^{1+\alpha}(\varepsilon,\omega), H^{\gamma_k+1}(\varepsilon,\omega)]_{\frac{\overline{\gamma_k}-\alpha}{\gamma_k-\alpha}},$$

we may estimate by the interpolation theorem ([19, Corollary 1.2.7])

$$\|v(\cdot,t)-v(\cdot,\tau)\|_{H^{\overline{\gamma_k}+1}(\varepsilon,\omega)} \le c\,\|v(\cdot,t)-v(\cdot,\tau)\|_{\mathcal{D}_\alpha}^{1-\frac{\overline{\gamma_k}-\alpha}{\gamma_k-\alpha}} \|v(\cdot,t)-v(\cdot,\tau)\|_{H^{\gamma_k+1}(\varepsilon,\omega)}^{\frac{\overline{\gamma_k}-\alpha}{\gamma_k-\alpha}},$$

where $c=c(\gamma_k,\overline{\gamma_k},\varepsilon)$. By Lemma 4.6, the second norm on the right hand side above is bounded on every compact interval contained in $(0,T]$, while the first tends to zero as $\tau \to t$ for $t,\tau \in [0,T]$.

In order to obtain the claim for $\partial^\alpha v_x$ we recall that by Theorem 4.2 we have $\partial^\alpha v_x \in C([0,T];L^2(0,1))$. Applying again the interpolation theorem we obtain for every $0<\varepsilon<\omega<1$, $0<\tau<t\le T$ and every $\alpha<\overline{\gamma_k}<\gamma_k<(k+1)\alpha$

$$\|\partial^\alpha v_x(\cdot,t)-\partial^\alpha v_x(\cdot,\tau)\|_{H^{\overline{\gamma_k}-\alpha}(\varepsilon,\omega)}$$

$$\le c(\gamma_k,\overline{\gamma_k},\alpha)\,\|\partial^\alpha v_x(\cdot,t)-\partial^\alpha v_x(\cdot,\tau)\|_{L^2(0,1)}^{1-\frac{\overline{\gamma_k}-\alpha}{\gamma_k-\alpha}} \|\partial^\alpha v_x(\cdot,t)-\partial^\alpha v_x(\cdot,\tau)\|_{H^{\gamma_k-\alpha}(\varepsilon,\omega)}^{\frac{\overline{\gamma_k}-\alpha}{\gamma_k-\alpha}}.$$

The first norm tends to zero as $\tau\to t$, while the second one is bounded on every compact interval contained in $(0,T]$ due to Lemma 4.6. This way we proved (4.60). The continuity of v_x in the case $\alpha\in(\frac{1}{2},1)$ follows by the Sobolev embedding from $v\in C([0,T],\mathcal{D}_\alpha)$. In the case $\alpha\in(0,\frac{1}{2}]$ we recall that $v_x\in C([0,T];L^2(0,1))$ and by (4.59) there exists $\gamma>\frac{1}{2}$ such that $v_x\in L^\infty_{loc}(0,T;{}_0H^\gamma(0,1))$. Hence, applying again an interpolation argument together with Sobolev embedding, we arrive at (4.61). □

Corollary 4.8. *Let us assume that $v_0\in\mathcal{D}_\alpha$. Let v be a solution to (4.7) given by Theorem 4.2. Then, for every $\alpha\in(0,1)$ there exists $\beta\in(\alpha,1)$ such that for every $0<\varepsilon<\omega<1$ there holds $v\in C((0,T];W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega))$.*

Proof. In the case $\alpha\in(0,\frac{1}{2})$ it is enough to notice that in view of Corollary 4.7 we have $v\in C((0,T];H^2(\varepsilon,\omega))$ for every $0<\varepsilon<\omega<1$. In the case $\alpha\in[\frac{1}{2},1)$ the claim follows from Corollary 4.7 by the Sobolev embedding. □

4.1.3. The existence and regularity of solutions to (4.2)

At last, we are ready to formulate and prove the result concerning a unique existence and regularity of solution to (4.2).

Theorem 4.9. *Let $b, T > 0$ and $\alpha \in (0,1)$. Let us assume that s satisfies (4.3). We further assume, that $u_0 \in H^{1+\alpha}(0,b)$, $u_0' \in {}_0H^{\alpha}(0,b)$ and $u_0(b) = 0$. Then, there exists a unique solution u to (4.2) such that $u, D^{\alpha}u \in C(\overline{Q_{s,T}})$, $u_t, \frac{\partial}{\partial x}D^{\alpha}u \in C(Q_{s,T})$ and for every $t \in [0,T]$ $u_x(\cdot,t) \in {}_0H^{\alpha}(0,s(t))$. Moreover, in the case $\alpha \in (\frac{1}{2},1)$ $u_x \in C(\overline{Q_{s,T}})$, while in the case $\alpha \in (0,\frac{1}{2}]$ $u_x \in C(\overline{Q_{s,T}} \setminus (\{t=0\} \times [0,b]))$. Furthermore, the boundary conditions $(4.2)_2$ are satisfied for every $t \in (0,T]$. Finally, there exists $\beta \in (\alpha,1)$ such that for every $t \in (0,T]$ and every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$.*

Proof. Firstly we will establish the results concerning the existence and regularity of solution to (4.7) and then, we will rewrite the results in terms of properties of solution to (4.2). We note that, under assumptions concerning regularity and traces of u_0 we obtain that v_0 defined in (4.6) belongs to $\mathcal{D}_{\alpha}$. Hence, there exists v a unique solution to (4.7) with the regularity given by Theorem 4.2, Lemma 4.3, Lemma 4.6 and Corollary 4.7. Since $v \in C([0,T];\mathcal{D}_{\alpha})$, by the Sobolev embedding we obtain that $v \in C([0,T]\times[0,1])$. Furthermore, from Corollary 4.8 we know that there exists $\beta \in (\alpha,1)$ such that $v \in C((0,T];W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega))$ for every $0 < \varepsilon < \omega < 1$.
We define the function u on $Q_{s,T}$ by the formula $u(x,t) = v(\frac{x}{s(t)},t)$. Since $v \in C([0,T] \times [0,1])$, we obtain that $u \in C(\overline{Q_{s,T}})$ and $v \in C((0,T];W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega))$ implies $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$ for every $t \in (0,T]$ and every $0 < \varepsilon < \omega < s(t)$. We note that $v_p(p,t) = s(t)u_x(x,t)$. Hence, from (4.61) we obtain that $u_x \in C(\overline{Q_{s,T}})$ in the case $\alpha \in (\frac{1}{2},1)$, for $\alpha \in (0,\frac{1}{2}]$ we get $u_x \in C(\overline{Q_{s,T}} \setminus (\{t=0\}\times[0,b]))$ and for $\alpha \in (0,1)$ we have $u_x(0,t) = 0$ for every $t \in (0,T]$. Furthermore,

$$u_t(x,t) = \frac{\partial}{\partial x}D^{\alpha}u(x,t) = \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial p}D^{\alpha}v(p,t) \text{ where } p = \frac{x}{s(t)}.$$

From Corollary 4.7 and the Sobolev embedding we may deduce that $\partial^{\alpha}v_p = \frac{\partial}{\partial p}D^{\alpha}v \in C((0,T]\times(0,1))$, which implies $\frac{\partial}{\partial x}D^{\alpha}u,\ u_t \in C(Q_{s,T})$. Finally, $v \in C([0,T];\mathcal{D}_{\alpha})$ implies that $u(t,s(t)) = 0$ and $u_x(\cdot,t) \in {}_0H^{\alpha}(0,s(t))$ for every $t \in [0,T]$. Hence, by (2.15) and Proposition 2.32 we obtain that for every $t \in [0,T]$ there holds $D^{\alpha}u(\cdot,t) \in {}_0H^1(0,s(t)) \subseteq AC[0,s(t)]$. Moreover, from the Sobolev embedding we infer that $D^{\alpha}v \in C([0,T]\times[0,1])$ and since $D^{\alpha}u(x,t) = \frac{1}{s^{\alpha}(t)}D^{\alpha}v(p,t)$ we obtain that $D^{\alpha}u \in C(\overline{Q_{s,T}})$. The uniqueness of solution follows from the energy estimate. If we assume that u with the regularity described above satisfies (4.2) with $u_0 \equiv 0$, then multiplying $(4.2)_1$ by u and integrating over $Q_{s,T}$ we arrive at

$$\int_0^T \int_0^{s(\tau)} u_t(x,\tau)\cdot u(x,\tau)dxd\tau - \int_0^T \int_0^{s(\tau)} \frac{\partial}{\partial x}D^{\alpha}u(x,\tau)\cdot u(x,\tau)dxd\tau = 0.$$

Applying the estimate (3.12) we get

$$\frac{1}{2}\int_0^T \int_0^{s(\tau)} \frac{d}{dt}\left|u(x,\tau)\right|^2 dxd\tau + c_{\alpha}\int_0^T \left\|u(\cdot,\tau)\right\|^2_{H^{\frac{1+\alpha}{2}(0,1)}(0,s(\tau))} d\tau \leq 0.$$

By the Fubini theorem we obtain that

$$\frac{1}{2}\int_0^{s(T)} |u(x,T)|^2\, dx + c_\alpha \int_0^T \|u(\cdot,\tau)\|^2_{H^{\frac{1+\alpha}{2}(0,1)}(0,s(\tau))}\, d\tau \leq 0$$

and hence $u \equiv 0$, which finishes the proof. □

4.2. A solution to Stefan problem

Before we prove the existence and uniqueness of the solution to Stefan problem, we need to derive the weak extremum principle for the system (4.2).

4.2.1. Extremum principles

We will begin with the auxiliary lemmas. Firstly, we will present an extended version of [13, Lemma 1] (see also [18, Theorem 1]).

Lemma 4.10. *Let us assume that $f : [0,L] \to \mathbb{R}$ is absolutely continuous on $[0,L]$ and for every $\varepsilon \in (0,L)$ it belongs to $W^{1,\frac{1}{1-\beta}}(\varepsilon, L)$ for some $\beta \in (0,1]$. Then for any $\alpha \in (0,\beta)$ $D^\alpha f$ is continuous on $(0,L)$ and*

1. *if f attains its local maximum at the point $x_0 \in (0,L]$, which is a global maximum on $[0,x_0]$, then for every $\alpha \in (0,\beta)$ there holds the inequality $(D^\alpha f)(x_0) \geq 0$. Furthermore, if f is not constant on $[0,x_0]$, then $(D^\alpha f)(x_0) > 0$.*
2. *If f attains its local minimum at the point $x_0 \in (0,L]$, which is a global minimum on $[0,x_0]$, then for every $\alpha \in (0,\beta)$ there holds the inequality $(D^\alpha f)(x_0) \leq 0$. Furthermore, if f is not constant on $[0,x_0]$, then $(D^\alpha f)(x_0) < 0$.*

Proof. Let us begin with the proof of the continuity of $D^\alpha f$. To this end, we fix $\alpha \in (0,\beta)$ and we take $x_1, x \in (0,L)$. Let us assume that $x_1 < x$. The case $x < x_1$ may be shown analogously. We note that

$$\Gamma(1-\alpha)\,|(D^\alpha f)(x) - (D^\alpha f)(x_1)| = \left|\int_0^x (x-p)^{-\alpha} f'(p)dp - \int_0^{x_1} (x_1-p)^{-\alpha} f'(p)dp\right|$$

$$\leq \int_{x_1}^x (x-p)^{-\alpha}\,|f'(p)|\, dp + \int_0^{x_1} [(x_1-p)^{-\alpha} - (x-p)^{-\alpha}]\,|f'(p)|\, dp.$$

The second term tends to zero as $x \to x_1$ because

$$\int_0^{x_1} (x-p)^{-\alpha}\,|f'(p)|\, dp \to \int_0^{x_1} (x_1-p)^{-\alpha}\,|f'(p)|\, dp$$

as $x \to x_1$ by the Lebesgue monotone convergence theorem. The first term also converges to zero because

$$\left|\int_{x_1}^x (x-p)^{-\alpha} f'(p)dp\right| \leq \|f'\|_{L^{\frac{1}{1-\beta}}(x_1,x)} \left(\int_{x_1}^x (x-p)^{-\frac{\alpha}{\beta}} dp\right)^\beta \to 0 \text{ as } x \to x_1.$$

Thus, the continuity of $D^\alpha f$ on $(0, L)$ is proven. Let us assume that f attains its local maximum at the point $x_0 \in (0, L]$, which is a global maximum on $[0, x_0]$. We define the function $g(x) := f(x_0) - f(x)$ for $x \in [0, L]$. We note that $g(x) \geq 0$ on $[0, x_0]$, $g(x_0) = 0$ and $(D^\alpha g)(x) = -(D^\alpha f)(x)$ for $x \in [0, L]$. For every $0 < \varepsilon < x \leq x_0$ we may estimate g as follows

$$g(x) \leq \int_x^{x_0} |g'(p)|\, dp \leq \|g'\|_{L^{\frac{1}{1-\beta}}(\varepsilon,L)} |x - x_0|^\beta . \tag{4.62}$$

Thus, for fixed $\alpha \in (0, \beta)$, applying the integration by parts formula, we get

$$(D^\alpha g)(x_0) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha} g'(p) dp$$

$$= \frac{1}{\Gamma(1-\alpha)} \lim_{p \to x_0^-} (x_0 - p)^{-\alpha} g(p) - \frac{x_0^{-\alpha} g(0)}{\Gamma(1-\alpha)} - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-1} g(p) dp.$$

From the estimate (4.62) we infer that the limit equals zero, hence

$$(D^\alpha g)(x_0) = -\frac{x_0^{-\alpha} g(0)}{\Gamma(1-\alpha)} - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-1} g(p) dp. \tag{4.63}$$

Thus $(D^\alpha g)(x_0) \leq 0$, which is equivalent with $(D^\alpha f)(x_0) \geq 0$. Furthermore, from the formula (4.63) we obtain that if f is not a constant function on $[0, x_0]$ then $(D^\alpha f)(x_0) > 0$. Substituting f by $-f$ we obtain the second part of the claim. $\square$

In the next lemma we will show that $\frac{\partial}{\partial x} D^\alpha f$ is non positive in the maximum point of f in the interior of the interval. This result, under stronger regularity assumptions, was proven in [22, Lemma 2.2]. Here we present the proof, where we do not demand C^2 regularity of f.

Lemma 4.11. *Let $f : [0, L] \to \mathbb{R}$ be an absolutely continuous function such that $f' \in W^{1,\frac{1}{1-\beta}}(\varepsilon, L)$ for every $\varepsilon > 0$ and for fixed $\beta \in (0, 1)$. Then for $\alpha \in (0, \beta)$ $\frac{\partial}{\partial x} D^\alpha f$ is continuous on $(0, L)$ and*

1. *if f attains its local maximum at $x_0 \in (0, L)$ which is a global maximum on $[0, x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) \leq 0$ for every $\alpha \in (0, \beta)$. Furthermore, if f is not constant on $[0, x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) < 0$.*
2. *If f attains its local minimum at $x_0 \in (0, L)$ which is a global minimum on $[0, x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) \geq 0$ for every $\alpha \in (0, \beta)$. Furthermore, if f is not constant on $[0, x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) > 0$.*

Proof. Let us begin with the proof of continuity of $\frac{\partial}{\partial x} D^\alpha f$. To this end, we fix $\alpha \in (0, \beta)$ and we take $x_1, x \in (0, L)$. Let us assume that $x_1 < x$. The case $x < x_1$ may be shown analogously. We note that for every $0 < \varepsilon < y < L$ there holds

$$\Gamma(1-\alpha)(\frac{\partial}{\partial x} D^\alpha f)(y) = \frac{\partial}{\partial y} \int_0^\varepsilon (y - p)^{-\alpha} f'(p) dp + \frac{\partial}{\partial y} \int_\varepsilon^y (y - p)^{-\alpha} f'(p) dp$$

$$= -\alpha \int_0^\varepsilon (y - p)^{-\alpha-1} f'(p) dp + \int_\varepsilon^y (y - p)^{-\alpha} f''(p) dp.$$

Hence, taking arbitrary $\varepsilon \in (0, x_1)$ we obtain that

$$\Gamma(1-\alpha)\left|\frac{\partial}{\partial x}D^\alpha f(x) - \frac{\partial}{\partial x}D^\alpha f(x_1)\right| \leq \alpha \int_0^\varepsilon [(x_1-p)^{-\alpha-1} - (x-p)^{-\alpha-1}]\,|f'(p)|\,dp$$
$$+\int_{x_1}^{x}(x-p)^{-\alpha}\,|f''(p)|\,dp + \int_\varepsilon^{x_1}[(x_1-p)^{-\alpha} - (x-p)^{-\alpha}]\,|f''(p)|\,dp.$$

The first term tends to zero as $x \to x_1$ because the convergence under the integral is uniform. The third term tends to zero because

$$\int_\varepsilon^{x_1}(x-p)^{-\alpha}\,|f''(p)|\,dp \to \int_\varepsilon^{x_1}(x_1-p)^{-\alpha}\,|f''(p)|\,dp$$

as $x \to x_1$ by the Lebesgue monotone convergence theorem. At last, the second term also converges to zero because

$$\left|\int_{x_1}^{x}(x-p)^{-\alpha}f''(p)dp\right| \leq \|f''\|_{L^{\frac{1}{1-\beta}}(x_1,x)}\left(\int_{x_1}^{x}(x-p)^{-\frac{\alpha}{\beta}}dp\right)^\beta \to 0 \text{ as } x \to x_1.$$

Thus, the continuity of $\frac{\partial}{\partial x}D^\alpha f$ on $(0, L)$ is proven. We will prove only the part of the claim concerning maximum, because the proof of the second part of the claim is analogous. We define $g(x) = f(x_0) - f(x)$. Then g is nonnegative on $[0, x_0]$, $g'(x_0) = 0$ and $\frac{\partial}{\partial x}D^\alpha g = -\frac{\partial}{\partial x}D^\alpha f$. We note that for every $0 < \varepsilon < x \leq x_0$ we may estimate

$$|g'(x)| \leq \int_x^{x_0}|g''(p)|\,dp \leq \|g''\|_{L^{\frac{1}{1-\beta}}(\varepsilon,L)}|x-x_0|^\beta \tag{4.64}$$

and

$$g(x) \leq \int_x^{x_0}|g'(p)|\,dp \leq \int_x^{x_0}\int_p^{x_0}|g''(r)|\,dr dp$$

$$\leq \|g''\|_{L^{\frac{1}{1-\beta}}(\varepsilon,L)}\int_x^{x_0}|p-x_0|^\beta\,dp = \|g''\|_{L^{\frac{1}{1-\beta}}(\varepsilon,L)}\frac{|x-x_0|^{\beta+1}}{\beta+1}. \tag{4.65}$$

Making use of these estimates we may differentiate under the integral sign as follows

$$(\frac{\partial}{\partial x}D^\alpha g)(x_0) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x}\int_0^x(x-p)^{-\alpha}g'(p)dp\right)(x_0)$$
$$= \frac{1}{\Gamma(1-\alpha)}\lim_{p\to x_0^-}(x_0-p)^{-\alpha}g'(p) - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{x_0}(x_0-p)^{-\alpha-1}g'(p)dp.$$

and the limit is equal to zero by the estimate (4.64). Applying integration by parts we obtain further

$$(\frac{\partial}{\partial x}D^\alpha g)(x_0) = -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{x_0}(x_0-p)^{-\alpha-1}g'(p)dp = -\frac{\alpha}{\Gamma(1-\alpha)}\lim_{p\to x_0^-}(x_0-p)^{-\alpha-1}g(p)$$
$$+\frac{\alpha}{\Gamma(1-\alpha)}x_0^{-\alpha-1}g(0) + \frac{\alpha(\alpha+1)}{\Gamma(1-\alpha)}\int_0^{x_0}(x_0-p)^{-\alpha-2}g(p)dp.$$

By (4.65) the limit equals zero, hence we arrive at

$$(\frac{\partial}{\partial x}D^\alpha g)(x_0) = \frac{\alpha}{\Gamma(1-\alpha)}x_0^{-\alpha-1}g(0) + \frac{\alpha(\alpha+1)}{\Gamma(1-\alpha)}\int_0^{x_0}(x_0-p)^{-\alpha-2}g(p)dp$$

and

$$(\frac{\partial}{\partial x}D^\alpha g)(x_0) \geq 0, \text{ which implies } (\frac{\partial}{\partial x}D^\alpha f)(x_0) \leq 0.$$

Furthermore, from the formula above, we obtain that if f is not a constant function on $[0, x_0]$ then $(\frac{\partial}{\partial x}D^\alpha f)(x_0) < 0$. □

Having proven Lemma 4.11, it is not difficult to deduce the weak extremum principle for parabolic-type problems involving $\frac{\partial}{\partial x}D^{\alpha}$.

Lemma 4.12 (Weak extremum principle). *Let s fulfills the assumption (4.3). We assume that u satisfies*

$$u_t - \frac{\partial}{\partial x}D^{\alpha}u = f \text{ in } Q_{s,T}$$

and has the following regularity $u \in C(\overline{Q_{s,T}})$, $u_t \in C(Q_{s,T})$, $u(\cdot,t) \in AC[0,s(t)]$ for every $t \in (0,T)$, $\frac{\partial}{\partial x}D^{\alpha}u \in C(Q_{s,T})$. Furthermore, for every $t \in (0,T)$, for every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$ for some $\beta \in (\alpha,1]$. Let us denote the parabolic boundary of $Q_{s,T}$ by $\partial\Gamma_{s,T} = \partial\overline{Q_{s,T}} \setminus (\{T\}\times(0,s(T)))$. Then,

1. *if $f \leq 0$, then u attains its maximum on $\partial\Gamma_{s,T}$.*
2. *If $f \geq 0$, then u attains its minimum on $\partial\Gamma_{s,T}$.*

Proof. The proof follows the standard argument for the linear parabolic equations. Firstly, we will prove the first part of the lemma. Let us assume that at some point $(x_0,t_0) \in \overline{Q_{s,T}} \setminus \partial\Gamma_{s,T}$ we have $u(x_0,t_0) > \max_{\partial\Gamma_{s,T}} u =: M$. We fix $\varepsilon > 0$ and we denote $v(x,t) = (u(x,t) - M)e^{-\varepsilon t}$ Then v attains its positive maximum in some point $(x_1,t_1) \in \overline{Q_{s,T}} \setminus \partial\Gamma_{s,T}$. We may calculate

$$v_t = u_t e^{-\varepsilon t} - \varepsilon v, \qquad \frac{\partial}{\partial x}D^{\alpha}v = e^{-\varepsilon t}\frac{\partial}{\partial x}D^{\alpha}u.$$

Thus

$$v_t - \frac{\partial}{\partial x}D^{\alpha}v = -\varepsilon v + f e^{-\varepsilon t}.$$

In particular

$$v_t(x_1,t_1) - \frac{\partial}{\partial x}D^{\alpha}v(x_1,t_1) = -\varepsilon v(x_1,t_1) + f(x_1,t_1)e^{-\varepsilon t_1} < 0.$$

Since (x_1,t_1) is a maximum point we have $v_t(x_1,t_1) \geq 0$ and by Lemma 4.11 we infer that $\frac{\partial}{\partial x}D^{\alpha}v(x_1,t_1) \leq 0$. Hence, $v_t(x_1,t_1) - \frac{\partial}{\partial x}D^{\alpha}v(x_1,t_1) \geq 0$, which leads to a contradiction. Setting $u := -u$ we obtain the second part of the claim. □

It is possible to relax the regularity assumptions in the statement above. We will also make use of the following version of the weak extremum principle.

Proposition 4.13. *[12] Let us assume that s fulfills the assumption (4.3), $u \in C(\overline{Q_{s,T}})$, $u(\cdot,t) \in AC[0,s(t)]$ for every $t \in (0,T)$, $\frac{\partial}{\partial x}D^{\alpha}u \in C(Q_{s,T})$, $u_t \in L^{\infty}(Q_{s,T})$ and for every $t \in (0,T)$, for every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$ for some $\beta \in (\alpha,1]$. If u satisfies*

$$u_t - \frac{\partial}{\partial x}D^{\alpha}u = f \text{ a.a. in } Q_{s,T},$$

where $f \leq 0$ a.a. on $Q_{s,T}$, then u attains its maximum on $\partial\Gamma_{s,T}$. In the case $f \geq 0$ a.a. on $Q_{s,T}$, u attains its minimum on $\partial\Gamma_{s,T}$.

Proof. We will consider only the case $f \leq 0$, because the case $f \geq 0$ is analogous. Let us assume that at some point $(x_0, t_0) \in \overline{Q_{s,T}} \setminus \partial\Gamma_{s,T}$ we have $u(x_0, t_0) > \max_{\partial\Gamma_{s,T}} u =: M$. We fix $\varepsilon > 0$ and we denote $v(x,t) = (u(x,t) - M)e^{-\varepsilon t}$. Then v attains its positive maximum in some point $(x_1, t_1) \in \overline{Q_{s,T}} \setminus \partial\Gamma_{s,T}$. We note that v satisfies

$$v_t(x,t) - \frac{\partial}{\partial x} D^\alpha v(x,t) = -\varepsilon v + f(x,t)e^{-\varepsilon t} \ \ a.e \quad \text{on} \quad Q_{s,T}.$$

Since $v(x_1, t_1) > 0$ and v is continuous we obtain that there exist $\delta > 0$ and $a > 0$ such that

$$\varepsilon v(x,t) \geq 2\delta \text{ for every } (x,t) \in [x_1 - a, x_1 + a] \times [t_1 - a, t_1].$$

Applying Lemma 4.11 we obtain that $-\frac{\partial}{\partial x} D^\alpha v(x_1, t_1) \geq 0$. By continuity of $\frac{\partial}{\partial x} D^\alpha v$ in $Q_{s,T}$ we infer that there exists $b \in (0, a)$ such that for every $b_1, b_2 \in (0, b)$ there holds

$$-\frac{\partial}{\partial x} D^\alpha v \geq -\delta \text{ on } [x_1 - b_1, x_1 + b_1] \times [t_1 - b_2, t_1].$$

Thus,

$$\varepsilon v - \frac{\partial}{\partial x} D^\alpha v \geq \delta \text{ on } [x_1 - b_1, x_1 + b_1] \times [t_1 - b_2, t_1].$$

We integrate this inequality on the cube $[x_1 - b_1, x_1 + b_1] \times [t_1 - b_2, t_1]$ and we arrive at

$$2\delta b_1 b_2 \leq \int_{x_1 - b_1}^{x_1 + b_1} \int_{t_1 - b_2}^{t_1} \varepsilon v(x,t) - \frac{\partial}{\partial x} D^\alpha v(x,t) dt dx = \int_{x_1 - b_1}^{x_1 + b_1} \int_{t_1 - b_2}^{t_1} -v_t(x,t) + f(x,t)e^{-\varepsilon t} dt dx.$$

Recalling that $f \leq 0$ a.a. on $Q_{s,T}$ we obtain that

$$2\delta b_1 b_2 \leq \int_{x_1 - b_1}^{x_1 + b_1} \int_{t_1 - b_2}^{t_1} -v_t(x,t) dt dx = \int_{x_1 - b_1}^{x_1 + b_1} v(x, t_1 - b_2) - v(x, t_1) dx.$$

We divide the inequality by $2b_1$ to get that

$$\delta b_2 \leq \frac{1}{2b_1} \int_{x_1 - b_1}^{x_1 + b_1} v(x, t_1 - b_2) - v(x, t_1) dx.$$

Passing with b_1 to zero we arrive at

$$\delta b_2 \leq v(x_1, t_1 - b_2) - v(x_1, t_1),$$

which is a contradiction with the fact that (x_1, t_1) is a maximum point of v. $\square$

4.2.2. Estimates

In the next two lemmas, we derive the bounds for the Caputo derivative of the solution to (4.2) and for the solution itself. This is a significant step in the proof of the existence of solution to (4.1).

Lemma 4.14. *Let us assume that the assumptions of Theorem 4.9 are satisfied and additionally $u_0 \geq 0$. Let u be a solution to (4.2) given by Theorem 4.9, then $(D^\alpha u)(s(t), t) \leq 0$. Furthermore, if $u_0 \not\equiv 0$, then for every $t \in (0, T]$ we have $(D^\alpha u)(s(t), t) < 0$.*

Proof. We note that by Theorem 4.9 we have $D^\alpha u \in C(\overline{Q_{s,T}})$, hence $D^\alpha u(s(t),t)$ is well defined and continuous on $[0,T]$. Furthermore, function u satisfies the assumptions of Lemma 4.12. Hence, it attains its minimum on the parabolic boundary. In order to show that the minimum is attained on the curve $(s(t),t)$ we introduce $u_\varepsilon = u - \varepsilon x$. Then u_ε satisfies

$$\begin{cases} u_{\varepsilon t} - \frac{\partial}{\partial x} D^\alpha u_\varepsilon = \frac{\varepsilon x^{-\alpha}}{\Gamma(1-\alpha)} & \text{in } Q_{s,T}, \\ u_{\varepsilon x}(0,t) = -\varepsilon, \;\; u_\varepsilon(s(t),t) = -\varepsilon s(t) & \text{for } t \in (0,T), \\ u_\varepsilon(x,0) = u_0(x) - \varepsilon x & \text{for } 0 < x < b. \end{cases}$$

From Lemma 4.12 we deduce that u_ε also attains its minimum on the parabolic boundary. Since $u_{\varepsilon,x}(0,t) < 0$ the minimum cannot be attained on the left boundary. Thus, we obtain that

$$u_\varepsilon(x,t) \geq \min\{u_0(x) - \varepsilon x, -\varepsilon s(t)\} \geq -\varepsilon s(t),$$

where we used the assumption $u_0 \geq 0$. Hence, $u(x,t) = u_\varepsilon(x,t) + \varepsilon x \geq -\varepsilon s(t)$. Passing to the limit with ε we obtain that $u \geq 0$. Hence, u attains its minimum, which is equal to zero, on the curve $(s(t),t)$. Applying the minimum principle in spatial dimension (Lemma 4.10), we obtain that $(D^\alpha u)(s(t),t) \leq 0$ for every $t \in [0,T]$.

It remains to show that if $u_0 \not\equiv 0$, then $(D^\alpha u)(s(t),t) < 0$ for every $t \in (0,T]$. Let us firstly establish the following lemma.

Lemma 4.15. *Let u be a nonnegative solution to $u_t - \frac{\partial}{\partial x} D^\alpha u = 0$ in $Q_{s,T}$, where s satisfies (4.3). We assume that u has the following regularity $u \in C(\overline{Q_{s,T}})$, $u_t \in C(Q_{s,T})$, $u(\cdot,t) \in AC[0,s(t)]$ for every $t \in (0,T)$, $\frac{\partial}{\partial x} D^\alpha u \in C(Q_{s,T})$. Furthermore, for every $t \in (0,T)$, for every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$ for some $\beta \in (\alpha,1]$. Let $t_0 \in (0,T]$ be fixed. Then if $u(s(t_0),t_0) = 0$, then either $(D^\alpha u)(s(t_0),t_0) < 0$ or $u \equiv 0$ on Q_{s,t_0}.*

Proof. In the proof we will employ the ideas introduced in [1, Appendix 2, Lemma 2.1]. We note that since u is nonnegative, u attains its minimum in $(s(t_0),t_0)$. Hence, by Lemma 4.10 we infer that either $(D^\alpha u)(s(t_0),t_0) < 0$ or $u(x,t_0) = 0$ for every $x \in [0,s(t_0)]$. We will show that the last condition leads to $u \equiv 0$ on Q_{s,t_0}. We will proceed by contradiction. Let us assume that $u \not\equiv 0$ on Q_{s,t_0}. Then, by continuity of u, we may choose $0 < t_1 < t_0$, $x_1 \in (0,s(t_1))$ and small $\delta > 0$, such that $u(x,t_1) > 0$ for every x belonging to $[x_1, x_1 + 2\delta]$. We introduce a nonnegative auxiliary function $\eta : [0, x_1 + 2\delta] \times [t_1, t_0] \to \mathbb{R}$ as follows

$$\eta(x,t) = \begin{cases} 0 & \text{on } [0,x_1] \times [t_1,t_0], \\ \varepsilon e^{-a(t-t_1)}[\delta^2 - (x - x_1 - \delta)^2]^2 & \text{on } (x_1, x_1 + 2\delta] \times [t_1,t_0], \end{cases}$$

where the constant $a > 0$ will be chosen later and $\varepsilon > 0$ is such that

$$\varepsilon[\delta^2 - (x - x_1 - \delta)^2]^2 \leq u(x,t_1) \text{ for every } x \in (x_1, x_1 + 2\delta).$$

Such a choice of $\varepsilon > 0$ is possible, if $\delta > 0$ is small, due to the continuity of u and the fact that $u(x_1, t_1) > 0$. Since $\eta(x_1, t) = \eta_x(x_1, t) = 0$, it is easy to notice that η satisfies regularity assumptions of Lemma 4.12 on $[0, x_1 + 2\delta] \times [t_1, t_0]$. Furthermore, we have

$$\eta(0,t) = 0, \ \ \eta(x_1 + 2\delta, t) = 0 \text{ for every } t \in [t_1, t_0]. \tag{4.66}$$

By the assumption concerning ε and the fact that u is nonnegative, there holds

$$\eta(x, t_1) \leq u(x, t_1) \text{ for every } x \in [0, x_1 + 2\delta]. \tag{4.67}$$

Our aim is to apply the weak minimum principle, obtained in Lemma 4.12, to the function $w := u - \eta$. To this end, we will show that for suitably chosen $a > 0$ we have

$$-\eta_t + \frac{\partial}{\partial x} D^\alpha \eta \geq 0 \text{ in } (0, x_1 + 2\delta) \times (t_1, t_0]. \tag{4.68}$$

At first we note that, by the definition of η we have

$$-\eta_t + \frac{\partial}{\partial x} D^\alpha \eta \equiv 0 \text{ on } (0, x_1] \times (t_1, t_0].$$

We note that for $x > x_1$ we may write

$$(\frac{\partial}{\partial x} D^\alpha \eta)(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_{x_1}^{x} (x-p)^{-\alpha} \eta_x(p,t) dp =: (\frac{\partial}{\partial x} D^\alpha_{x_1} \eta)(x,t).$$

In order to calculate $\frac{\partial}{\partial x} D^\alpha_{x_1} \eta$ we note that $\eta_x(x_1, t) = 0$, thus $\frac{\partial}{\partial x} D^\alpha_{x_1} \eta = D^\alpha_{x_1} \eta_x$. Let us perform the calculations. We have

$$\eta_x(x,t) = -4\varepsilon e^{-a(t-t_1)} [\delta^2 - (x - x_1 - \delta)^2](x - x_1 - \delta)$$

and

$$\eta_{xx}(x,t) = -4\varepsilon e^{-a(t-t_1)} (\delta^2 - 3(x - x_1 - \delta)^2).$$

Thus, we may write

$$\frac{\partial}{\partial x} D^\alpha_{x_1} \eta = \frac{4\varepsilon e^{-a(t-t_1)}}{\Gamma(1-\alpha)} \left[3 \int_{x_1}^{x} (x-p)^{-\alpha} (p - x_1 - \delta)^2 dp - \delta^2 \int_{x_1}^{x} (x-p)^{-\alpha} dp \right].$$

Calculating the last integral we obtain, that for $(x,t) \in (x_1, x_1 + 2\delta) \times (t_1, t_0)$ there holds

$$-\eta_t + \frac{\partial}{\partial x} D^\alpha \eta =$$

$$\varepsilon e^{-a(t-t_1)} \left(a[\delta^2 - (x - x_1 - \delta)^2]^2 + \frac{4}{\Gamma(1-\alpha)} \left[3 \int_{x_1}^{x} \frac{(p - x_1 - \delta)^2}{(x-p)^\alpha} dp - \frac{\delta^2 (x - x_1)^{1-\alpha}}{1-\alpha} \right] \right). \tag{4.69}$$

We will show that the last expression is nonnegative for every $(x,t) \in (x_1, x_1 + 2\delta) \times (t_1, t_0)$ for suitably chosen $a > 0$. At first, we note that

$$\kappa_\alpha := \frac{1}{2-\alpha} \left[3 - \sqrt{3} \sqrt{\frac{1+\alpha}{3-\alpha}} \right] > 1 \text{ for every } \alpha \in (0,1). \tag{4.70}$$

Let us introduce

$$\omega_{\alpha,\delta} := \frac{2\delta(\kappa_\alpha - 1)}{\kappa_\alpha}. \tag{4.71}$$

We will consider three cases.

1. Let $x \in [x_1 + \frac{1}{3}\delta, x_1 + 2\delta - \omega_{\alpha,\delta}]$. Then,

$$[\delta^2 - (x - x_1 - \delta)^2]^2 \geq [\delta^2 - (\delta - \omega_{\alpha,\delta})^2]^2 \text{ and } (x - x_1)^{1-\alpha} \leq (2\delta - \omega_{\alpha,\delta})^{1-\alpha}.$$

Thus, for $a \geq \frac{4\delta^2}{\Gamma(2-\alpha)} \frac{(2\delta-\omega_{\alpha,\delta})^{1-\alpha}}{[\delta^2-(\delta-\omega_{\alpha,\delta})^2]^2}$ we have

$$a[\delta^2 - (x - x_1 - \delta)^2]^2 \geq \frac{4\delta^2 (x - x_1)^{1-\alpha}}{\Gamma(2 - \alpha)}$$

and the expression (4.69) is nonnegative.

2. If $x \in (x_1, x_1 + \frac{1}{3}\delta]$, we may notice that $(x - (x_1 + \delta))^2 \geq \frac{4}{9}\delta^2$ and thus

$$3 \int_{x_1}^{x} (x-p)^{-\alpha}(p - x_1 - \delta)^2 dp \geq 3\frac{4\delta^2}{9} \int_{x_1}^{x} (x-p)^{-\alpha} dp = \frac{4\delta^2}{3} \frac{(x - x_1)^{1-\alpha}}{1 - \alpha},$$

which ensures that (4.69) is nonnegative.

3. It remains to deal with the case $x \in [x_1 + 2\delta - \omega_{\alpha,\delta}, x_1 + 2\delta)$. We apply the substitution $p = x_1 + r(x - x_1)$ to obtain that

$$3 \int_{x_1}^{x} (x-p)^{-\alpha}(p - x_1 - \delta)^2 dp = 3 \int_{0}^{1} (1-r)^{-\alpha}(r(x - x_1) - \delta)^2 dr (x - x_1)^{1-\alpha}.$$

Thus, it is enough to prove that for each $x \in [x_1 + 2\delta - \omega_{\alpha,\delta}, x_1 + 2\delta]$

$$3 \int_{0}^{1} (1-r)^{-\alpha}(r(x - x_1) - \delta)^2 dr \geq \frac{\delta^2}{1 - \alpha},$$

which is equivalent with

$$3 \int_{0}^{1} (1-r)^{-\alpha} r^2 dr (x - x_1)^2 - 6\delta \int_{0}^{1} (1-r)^{-\alpha} r dr (x - x_1) + \frac{2\delta^2}{1 - \alpha} \geq 0.$$

Calculating the above integrals and dividing the inequality by 2 we have

$$\frac{\delta^2}{1-\alpha} - 3\delta(x - x_1)\frac{\Gamma(1-\alpha)}{\Gamma(3-\alpha)} + 3(x - x_1)^2 \frac{\Gamma(1-\alpha)}{\Gamma(4-\alpha)} \geq 0.$$

Multiplying this inequality by $\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$ we obtain another equivalent inequality

$$\delta^2 - \frac{3(x - x_1)}{2 - \alpha}\delta + \frac{3(x - x_1)^2}{(2 - \alpha)(3 - \alpha)} \geq 0.$$

By direct calculations we see that the roots of the function

$$f(\delta) := \delta^2 - \frac{3(x - x_1)}{2 - \alpha}\delta + \frac{3(x - x_1)^2}{(2 - \alpha)(3 - \alpha)}$$

are given by the formula

$$\delta_{\mp} = \frac{(x - x_1)}{2(2 - \alpha)} \left[3 \mp \sqrt{3} \sqrt{\frac{1 + \alpha}{3 - \alpha}} \right].$$

Thus, it is enough to show that $\delta \leq \delta_-$ for every choice of $x \in [x_1 + 2\delta - \omega_{\alpha,\delta}, x_1 + 2\delta]$. Recalling the definitions (4.70) and (4.71), we have

$$\delta_- = \kappa_\alpha \frac{(x - x_1)}{2} \geq \kappa_\alpha \frac{2\delta - \omega_{\alpha,\delta}}{2} = \delta.$$

In this way we have shown that (4.69) is nonnegative for $x \in [x_1 + 2\delta - \omega_{\alpha,\delta}, x_1 + 2\delta)$. Summing up the result, we obtained that (4.69) is nonnegative for every $x \in (x_1, x_1 + 2\delta)$ and, as a consequence, (4.68) holds.

Let us define $w = u - \eta$. Then, applying (4.66), (4.67), (4.68) we obtain that

$$\begin{cases} w_t - \frac{\partial}{\partial x} D^{\alpha}_{x_1} w \geq 0 & \text{in } (0, x_1 + 2\delta) \times (t_1, t_0], \\ w(0,t) = u(0,t) \geq 0, \quad w(x_1 + 2\delta, t) = u(x_1 + 2\delta, t) \geq 0 & \text{for } t \in [t_1, t_0], \\ w(x, t_1) \geq 0 & \text{for } x \in [0, x_1 + 2\delta]. \end{cases}$$

Obviously Lemma 4.12 is true also if we consider a problem in a cylindrical domain, thus we may apply the minimum principle, to obtain that w attains its minimum on the parabolic boundary of $[0, x_1 + 2\delta] \times [t_1, t_0]$. Thus, $w \geq 0$ in $[0, x_1 + 2\delta] \times [t_1, t_0]$. In particular

$$u(x, t_0) \geq \eta(x, t_0) = \varepsilon e^{-a(t_0 - t_1)}[\delta^2 - (x - x_1 - \delta)^2]^2 > 0 \text{ for every } x \in (x_1, x_1 + 2\delta).$$

This is a contradiction with $u(x, t_0) = 0$ on $[0, s(t_0)]$. Thus, we obtained that $u \equiv 0$ in Q_{s,t_0}. ☐

From Lemma 4.15 we infer that $D^\alpha u(s(t_0), t_0) < 0$, because otherwise we obtain a contradiction with $u_0 \not\equiv 0$ and the continuity of u. In this way we proved Lemma 4.14. ☐

Lemma 4.16. *Let us assume that $u_0 \geq 0$ satisfies the assumptions of Theorem 4.9. We assume further that there exists $M > 0$ such that*

$$u_0(x) \leq \frac{M\Gamma(2-\alpha)}{b^{1-\alpha}}(b - x) \text{ for every } x \in [0, b]. \tag{4.72}$$

Moreover, let s fulfill the assumption (4.3), where the constant M comes from (4.72). If u is a solution to (4.2) given by Theorem 4.9, then there hold the following bounds

$$(D^\alpha u)(s(t), t) \geq -M \text{ for every } t \in (0, T) \tag{4.73}$$

and

$$0 \leq u(x, t) \leq M\Gamma(2-\alpha)s^{\alpha-1}(t)(s(t) - x) \text{ for } (x, t) \in Q_{s,T}. \tag{4.74}$$

Remark 4.2. *We note that in the case $\alpha \in (\frac{1}{2}, 1)$ the assumption (4.72) is trivial, since from $u_0 \in H^{1+\alpha}(0, 1)$ follows that u_0 is Lipschitz continuous.*

Proof. In the proof we follow the ideas introduced in [1, Proposition 4.2], where the author consider the classical Stefan problem. Let us denote by u a solution to (4.2) given by Theorem 4.9. We define an auxiliary function v by the formula

$$v(x, t) = M_0 s^{\alpha-1}(t)(s(t) - x),$$

where $M_0 = M\Gamma(2 - \alpha)$. Then we may calculate

$$(D^\alpha v)(s(t), t) = -\frac{M_0 s^{\alpha-1}(t)}{\Gamma(1-\alpha)} \int_0^{s(t)} (s(t) - p)^{-\alpha} dp = -\frac{M_0}{\Gamma(2-\alpha)} = -M.$$

Moreover, making use of (4.72) we obtain

$$v(s(t), t) = 0, \quad v_x(x, t) = -M_0 s^{\alpha-1}(t) < 0 = u_x(0, t), \quad v(x, 0) = \frac{M_0}{b^{1-\alpha}}(b - x) \geq u_0(x).$$

We may calculate further

$$v_t(x, t) = M_0 \alpha s^{\alpha-1}(t)\dot{s}(t) + (1 - \alpha)M_0 s^{\alpha-2}(t)\dot{s}(t)x,$$
$$\frac{\partial}{\partial x} D^\alpha v(x, t) = -\frac{M_0 s^{\alpha-1}(t)}{\Gamma(1-\alpha)} x^{-\alpha}.$$

Together we have

$$v_t(x, t) - \frac{\partial}{\partial x} D^\alpha v(x, t)$$
$$= M_0 \alpha s^{\alpha-1}(t)\dot{s}(t) + (1 - \alpha)M_0 s^{\alpha-2}(t)\dot{s}(t)x + \frac{M_0 s^{\alpha-1}(t)}{\Gamma(1-\alpha)} x^{-\alpha} =: -f(x, t) \geq 0.$$

We define the function $w = u - v$. Then w satisfies

$$\begin{cases} w_t - \frac{\partial}{\partial x} D^\alpha w = f & \text{in } Q_{s,T}, \\ w_x(0,t) > 0, \quad w(s(t),t) = 0 & \text{for } t \in (0,T), \\ w(x,0) \leq 0 & \text{for } 0 < x < s(0). \end{cases}$$

Since the function s is Lipschitz continuous, we get that $w_t \in L^\infty(Q_{s,T})$. Thus, we may apply the weak maximum principle from Proposition 4.13, to function w in order to obtain that $\max_{\overline{Q_{s,T}}} w = \max_{\partial\Gamma_{s,T}} w$. Since $w_x(0,t) > 0$ maximum cannot be attained on the left boundary. We note that $w(x,0) \leq 0$ and $w(s(t),t) = 0$, thus $w \leq 0$ and we obtain (4.74). Moreover, w needs to admit its maximum on the part of the boundary $(s(t),t)$, where it is equal to zero. Hence, by Lemma 4.10, we get $(D^\alpha w)(s(t),t) \geq 0$, thus $(D^\alpha u)(s(t),t) \geq (D^\alpha v)(s(t),t) \;=\; -M$. □

4.2.3. A proof of the final result

Finally, we are ready to prove the Theorem 4.1. At first, we will show the existence of a solution. The method of the proof relays on the construction of the free boundary $s(\cdot)$ by the Schauder fixed point theorem. Subsequently we show that the obtained solution is unique. It will be done by proving the monotone dependence of solutions upon data.

Theorem 4.17. *Let $b, T > 0$ and $\alpha \in (0,1)$. Let us assume that $u_0 \in H^{1+\alpha}(0,b)$, $u_0' \in {}_0H^\alpha(0,b)$, $u_0(b) = 0$ and $u_0 \geq 0$, $u_0 \not\equiv 0$. Further let us assume that there exists $M > 0$ such that for every $x \in [0,b]$*

$$u_0(x) \leq \frac{M\Gamma(2-\alpha)}{b^{1-\alpha}}(b-x).$$

Then, there exists (u,s) a solution to (4.1), such that $s \in C^1([0,T])$, for every $t \in [0,T]$ there holds $0 < \dot{s}(t) \leq M$, $u \in C(\overline{Q_{s,T}}), D^\alpha u \in C(\overline{Q_{s,T}})$, $u_t, \frac{\partial}{\partial x} D^\alpha u \in C(Q_{s,T})$ and for every $t \in [0,T]$ $u_x(\cdot,t) \in {}_0H^\alpha(0,s(t))$. Moreover, in the case $\alpha \in (\frac{1}{2},1)$ we have $u_x \in C(\overline{Q_{s,T}})$, while in the case $\alpha \in (0,\frac{1}{2}]$ we have $u_x \in C(\overline{Q_{s,T}} \setminus (\{t=0\} \times [0,b]))$. Furthermore, the boundary conditions $(4.1)_2$ are satisfied for every $t \in [0,T]$. Finally, there exists $\beta \in (\alpha,1)$, such that for every $t \in (0,T]$ and every $0 < \varepsilon < \omega < s(t)$ there holds $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$.

Proof. We follow the idea introduced in the proof of [1, Theorem 5.1]. We define the set

$$\Sigma := \{s \in C^{0,1}[0,T], \;\; 0 < \dot{s} \leq M, \;\; s(0) = b\}.$$

Then for every $s \in \Sigma$ there exists a unique solution to (4.2), given by Theorem 4.9. We will show that Σ is a compact and convex subset of a Banach space $C([0,T])$ with a maximum norm. The convexity of Σ is straightforward. In order to show that Σ is compact we will firstly show that it is closed in $C([0,T])$. Let us denote by $\{s_k\}$ the sequence in Σ which is convergent in $C([0,T])$ to some s. Then $s \in C([0,T])$ and $s(0) = b$. Moreover, for every $k \in \mathbb{N}$ and every $\tau, t \in [0,T]$ we have

$$|s_k(t) - s_k(\tau)| \leq M \, |t - \tau| \, .$$

Passing to the limit with k we arrive at

$$|s(t) - s(\tau)| \leq M \, |t - \tau|$$

and hence $s \in \Sigma$. We note that any sequence in Σ which is bounded in $C([0,T])$ is equicontinuous, thus we may apply Arzela-Ascoli theorem to obtain that Σ is relatively compact in $C([0,T])$. We have already proven that Σ is closed, hence we obtain that Σ is compact in $C([0,T])$. For $s \in \Sigma$ we define the operator

$$(Ps)(t) = b - \int_0^t (D^\alpha u)(s(\tau), \tau) d\tau,$$

where u is a solution to (4.2), corresponding to s, given by Theorem 4.9. We would like to apply the Schauder fixed point theorem ([8, Theorem 3, Chapter 9.2.2.]), thus we have to show that $P : \Sigma \to \Sigma$ and that it is continuous in maximum norm. Clearly we have $(Ps)(0) = b$ and from Lemma 4.14 and estimate (4.73) we infer

$$0 < \frac{d}{dt}(Ps)(t) = -(D^\alpha u)(s(t), t) \leq M.$$

Hence, $P : \Sigma \to \Sigma$.
To prove that P is continuous in maximum norm, we firstly note that integrating the first equation in (4.2) we obtain

$$(D^\alpha u)(s(\tau), \tau) = \int_0^{s(\tau)} u_t(x, \tau) dx,$$

where we made use of the fact that for every fixed $t > 0$ $u_x(x,t)$ is bounded and hence $(D^\alpha u)(0,t) = 0$. Thus, we may rewrite the formula for P as follows

$$(Ps)(t) = b - \int_0^t \int_0^{s(\tau)} u_t(x,\tau) dx d\tau = b - \int_0^b \int_0^t u_t(x,\tau) d\tau dx - \int_b^{s(t)} \int_{s^{-1}(x)}^t u_t(x,\tau) d\tau dx$$

$$= b - \int_0^b u(x,t)dx + \int_0^b u(x,0)dx - \int_b^{s(t)} u(x,t)dx = b + \int_0^b u_0(x)dx - \int_0^{s(t)} u(x,t)dx. \quad (4.75)$$

Now, we take arbitrary s_1, $s_2 \in \Sigma$. Let us define $s_{min}(t) = \min\{s_1(t), s_2(t)\}$, $s_{max}(t) = \max\{s_1(t), s_2(t)\}$. We also define function $i = i(t) = 1$ if $s_{max}(t) = s_1(t)$ and $i = 2$ otherwise. Let u_1 and u_2 be two solutions to (4.2), given by Theorem 4.9, corresponding to s_1 and s_2 respectively. Let us define $v(x,t) = u_1(x,t) - u_2(x,t)$ and $v^\varepsilon(x,t) = v(x,t) + \varepsilon x$. Then v^ε satisfies

$$\begin{cases} v^\varepsilon_t - \frac{\partial}{\partial x} D^\alpha v^\varepsilon = -\frac{\varepsilon x^{-\alpha}}{\Gamma(1-\alpha)} & \text{in } \{(x,t) : 0 < x < s_{min}(t), 0 < t < T\} =: Q_{s_{min},T}, \\ v^\varepsilon_x(0,t) = \varepsilon, & \text{for } t \in (0,T), \\ v^\varepsilon(x,0) = \varepsilon x & \text{in } 0 < x < b. \end{cases}$$

From Lemma 4.12 we obtain that v^ε attains its maximum on the parabolic boundary. We may estimate

$$|v^\varepsilon(s_{min}(t),t)| \le |u_1(s_{min}(t),t)| + |u_2(s_{min}(t),t)| + \varepsilon s_{min}(T) = |u_i(s_{min}(t),t)| + \varepsilon s_{min}(T)$$

and since $v^\varepsilon(x,0) = \varepsilon x \le \varepsilon s_{min}(T)$ and $v^\varepsilon_x(0,t) > 0$ we obtain that

$$\max_{Q_{s_{min},T}} v^\varepsilon \le |u_i(s_{min}(t),t)| + \varepsilon s_{min}(T).$$

Let us denote $M_0 := M\Gamma(2-\alpha)$. Then, applying the estimate (4.74) from Lemma 4.16 we get

$$|u_i(s_{min}(t),t)| \le M_0 s_{max}^{\alpha-1}(t)(s_{max}(t) - s_{min}(t)) \le M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)|.$$

Hence,

$$\max_{Q_{s_{min},T}} v = \max_{Q_{s_{min},T}} (v^\varepsilon - \varepsilon x) \le M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| + \varepsilon s_{min}(T).$$

Passing with ε to zero we obtain

$$\max_{Q_{s_{min},T}} v \le M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)|.$$

To estimate v from below we proceed similarly. We introduce $v_\varepsilon(x,t) = v(x,t) - \varepsilon x$. Then v_ε satisfies

$$\begin{cases} v_{\varepsilon t} - \frac{\partial}{\partial x} D^\alpha v_\varepsilon = \frac{\varepsilon x^{-\alpha}}{\Gamma(1-\alpha)} & \text{in } Q_{s_{min},T} \\ v_{\varepsilon x}(0,t) = -\varepsilon, & \text{for } t \in (0,T) \\ v_\varepsilon(x,0) = -\varepsilon x & \text{in } 0 < x < b. \end{cases}$$

Lemma 4.12 implies that v_ε attains its minimum on the parabolic boundary. We may estimate

$$v_\varepsilon(s_{min}(t),t) \ge -|u_i(s_{min}(t),t)| - \varepsilon s_{min}(T)$$

and since $v_\varepsilon(x,0) = -\varepsilon x \ge -\varepsilon s_{min}(T)$ and $v_{\varepsilon x}(0,t) < 0$ we obtain that

$$\min_{Q_{s_{min},T}} v_\varepsilon \ge -|u_i(s_{min}(t),t)| - \varepsilon s_{min}(T) \ge -M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| - \varepsilon s_{min}(T),$$

thus

$$\min_{Q_{s_{min},T}} v = \min_{Q_{s_{min},T}} (v_\varepsilon + \varepsilon x) \ge -M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| - \varepsilon s_{min}(T).$$

Passing to the limit with ε we arrive at

$$\min_{Q_{s_{min},T}} v \geq -M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| .$$

Combining the estimates for minimal and maximal value of v we obtain

$$\max_{Q_{s_{min},T}} |v| \leq M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| .$$

Furthermore, estimate (4.74) implies that

$$\int_{s_{min}(t)}^{s_{max}(t)} u_i(x,t)dx \leq M_0 s_{\max}^{\alpha-1}(t) \int_{s_{min}(t)}^{s_{max}(t)} (s_{max}(t) - x)dx \leq M_0 b^{\alpha-1}(s_{max}(t) - s_{min}(t))^2.$$

Finally, we may estimate

$$\begin{aligned} |(Ps_2)(t) - (Ps_1)(t)| &= \left| \int_0^{s_2(t)} u_2(x,t)dx - \int_0^{s_1(t)} u_1(x,t)dx \right| \\ &\leq \int_0^{s_{min}(t)} |v(x,t)|\, dx + \int_{s_{min}(t)}^{s_{max}(t)} u_i(x,t)dx \\ &\leq s_{min}(t) \max_{Q_{s_{min},T}} |v| + (s_{max}(t) - s_{min}(t))^2 M_0 b^{\alpha-1} \\ &\leq (b+MT)M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)| + M_0 b^{\alpha-1} \max_{\tau\in[0,t]} |s_1(\tau) - s_2(\tau)|^2 . \end{aligned}$$

Thus P is continuous and by the Schauder fixed point theorem there exist a fixed point of P. In this way we have proven that there exists a pair (u,s) that satisfies the system (4.1), where $s \in \Sigma$ and u is given by Theorem 4.9. We note that $\dot{s}(t) = -D^\alpha u(s(t),t)$ and since $D^\alpha u \in C(\overline{Q_{s,T}})$ we obtain that $\dot{s} \in C[0,T]$. This finishes the proof. □

In order to show that the obtained solution is unique we will prove the monotone dependence upon data.

Theorem 4.18. *Let (u_i, s_i) be a solution to (4.1) given by Theorem 4.17 corresponding to b_i and u_0^i for $i = 1,2$. If $b_1 \leq b_2$ and $u_0^1 \leq u_0^2$, then for every $t \in [0,T]$ we have $s_1(t) \leq s_2(t)$.*

Proof. In the proof we apply the ideas introduced in [1]. We divide the proof into two steps.

1. Let us firstly discuss the case $b_1 < b_2$, $u_0^1 \leq u_0^2$ and $u_0^1 \not\equiv u_0^2$ on $[0,b_1]$. We will proceed by contradiction. Let us assume that there exists $t \in [0,T]$ such that $s_1(t) > s_2(t)$. We denote $t_0 = \inf\{t \in [0,T] : s_1(t) = s_2(t)\}$. Then by virtue of weak minimum principle (Lemma 4.12) function $v = u_2 - u_1$ is nonnegative on Q_{s_1,t_0} and $v(s_1(t_0),t_0) = 0$. Thus, from Lemma 4.15 we infer that either $v \equiv 0$ on Q_{s_1,t_0} or $(D^\alpha v)(s(t_0),t_0) < 0$. The first possibility is a contradiction with $u_0^1 \not\equiv u_0^2$. Hence,

$$0 > (D^\alpha v)(s(t_0),t_0) = (D^\alpha u_2)(s(t_0),t_0) - (D^\alpha u_1)(s(t_0),t_0) = \dot{s_1}(t_0) - \dot{s_2}(t_0)$$

and we obtain the contradiction with the definition of t_0. Thus, we obtain that if $b_1 < b_2$, $u_0^1 \le u_0^2$ and $u_0^1 \not\equiv u_0^2$ on $[0, b_1]$, then $s_1(t) \le s_2(t)$ for every $t \in [0, T]$.
2. In the general case, that is $b_1 \le b_2$ and $u_0^1 \le u_0^2$ we proceed as follows. We fix $\delta > 0$ and denote by u_0^δ a smooth function defined on $[0, b_2 + \delta]$ in such a way that $u_0^\delta \equiv 0$ on $[b_2 + \delta/2, b_2 + \delta]$, $u_0^\delta \ge u_0^2$ on $[0, b_2]$ and $\max_{x \in [0,b_2]}(u_0^\delta(x) - u_0^2(x)) = \delta$, $\max_{x\in[b_2,b_2+\delta/2]} u_0^\delta(x) \le \delta$. Then, we denote by (u_δ, s_δ) the solution to (4.1) given by Theorem 4.1 corresponding to u_0^δ. By the first step of the proof, we have $s_1 \le s_\delta$ and $s_2 \le s_\delta$. On the other hand performing calculations as in (4.75) we have

$$\begin{aligned} s_\delta(t) &= b_2 + \delta + \int_0^t \dot{s_\delta}(\tau)d\tau = b_2 + \delta - \int_0^t (D^\alpha u_\delta)(s_\delta(\tau), \tau)d\tau \\ &= b_2 + \delta + \int_0^{b_2+\delta} u_0^\delta(x)dx - \int_0^{s_\delta(t)} u_\delta(x,t)dx \end{aligned}$$

and

$$s_2(t) = b_2 + \int_0^{b_2} u_0^2(x)dx - \int_0^{s_2(t)} u_2(x,t)dx.$$

Subtracting these identities we obtain

$$s_\delta(t) - s_2(t) = \delta + \int_0^{b_2+\delta} u_0^\delta(x)dx - \int_0^{b_2} u_0^2(x)dx - \int_0^{s_\delta(t)} u_\delta(x,t)dx + \int_0^{s_2(t)} u_2(x,t)dx$$

$$= \delta + \int_0^{b_2} u_0^\delta(x) - u_0^2(x)dx + \int_{b_2}^{b_2+\frac{\delta}{2}} u_0^\delta(x)dx - \int_{s_2(t)}^{s_\delta(t)} u_\delta(x,t)dx - \int_0^{s_2(t)} [u_\delta(x,t) - u_2(x,t)]dx.$$

The last two integrals are positive due to Lemma 4.12. Making use of $\left\| u_0^\delta - u_0^2 \right\|_{L^\infty(0,b_2)} = \delta$ we obtain

$$s_1(t) \le s_\delta(t) \le s_2(t) + \delta + b_2\delta + \frac{\delta}{2}\delta \text{ for every } t \in [0, T].$$

Passing to the limit with δ we obtain that $s_1(t) \le s_2(t)$ for every $t \in [0, T]$. □

Corollary 4.19. *From Theorem 4.18 applied together with Theorem 4.9 it follows that the solution (u, s) to problem (4.1) given by Theorem 4.17 is unique. This finishes the proof of Theorem 4.1.*

4.3. A self-similar solution

In this section we will find a special solution to space-fractional Stefan problem. It is worth to mention that the self-similar solution to space-fractional Stefan problem was obtained independently in the recent paper [26]. Let us discuss the following system

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } \{(x,t) : 0 < x < s(t),\ 0 < t < \infty\}, \\ u(0,t) = c_1, \quad u(t, s(t)) = 0 & \text{for } t \in (0, \infty), \\ \dot{s}(t) = -(D^\alpha u)(s(t), t) & \text{for } t \in (0, \infty), \end{cases} \tag{4.76}$$

where we assume that $s(0) = 0$ and $c_1 > 0$. We would like to find a scale-invariant solution to this problem. In order to find the appropriate scaling we introduce

$$u^\lambda(x,t) := \lambda^c u(\lambda^a x, \lambda^b t) \quad \text{for} \quad a, b, c, \lambda > 0.$$

Let us perform the calculations

$$u_t(\lambda^a x, \lambda^b t) = \lambda^{-b-c} u_t^\lambda(x,t) \text{ and } u_x(\lambda^a x, \lambda^b t) = \lambda^{-a-c} u_x^\lambda(x,t).$$

Further we have,

$$\Gamma(1-\alpha)\frac{\partial}{\partial x} D^\alpha u^\lambda(x,t) = \frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha} u_x^\lambda(p,t)dp = \frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}\lambda^{a+c} u_x(\lambda^a p, \lambda^b t)dp.$$

Applying the substitution $\lambda^a p = w$ we obtain

$$\Gamma(1-\alpha)\partial^\alpha u_x^\lambda(x,t) = \lambda^c \frac{\partial}{\partial x}\int_0^{\lambda^a x} (x - w\lambda^{-a})^{-\alpha} u_x(w, \lambda^b t)dw$$

$$= \lambda^{a\alpha+c}\frac{\partial}{\partial x}\int_0^{\lambda^a x} (\lambda^a x - w)^{-\alpha} u_x(w,\lambda^b t)dw = \lambda^{a(\alpha+1)+c}(\frac{\partial}{\partial x} D^\alpha u)(\lambda^a x, \lambda^b t).$$

Hence, if u satisfies $(4.76)_1$, then

$$0 = \lambda^{-c}\lambda^{-b} u_t^\lambda(x,t) - \lambda^{-c}\lambda^{-a(\alpha+1)}\frac{\partial}{\partial x} D^\alpha u^\lambda(x,t).$$

We are looking for a self-similar solution, so if we suppose that $u \equiv u^\lambda$ we arrive at

$$b = a(\alpha+1) \quad \text{and} \quad c = 0.$$

Motivated by the above calculation, we introduce the similarity variable $\xi = xt^{-\frac{1}{\alpha+1}}$ and we define

$$F(\xi) = F(xt^{-\frac{1}{\alpha+1}}) := u(x,t).$$

Let us rewrite the equation $(4.76)_1$ in terms of function F. We may calculate as follows

$$u_t(x,t) = -\frac{1}{\alpha+1} x t^{-\frac{1}{\alpha+1}-1} F'(\xi), \;\; u_x(x,t) = t^{-\frac{1}{\alpha+1}} F'(\xi) \tag{4.77}$$

and

$$\Gamma(1-\alpha)\frac{\partial}{\partial x} D^\alpha u(x,t) = t^{-\frac{1}{\alpha+1}}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha} F'(pt^{-\frac{1}{\alpha+1}})dp$$

$$= \frac{\partial}{\partial x}\int_0^{xt^{-\frac{1}{1+\alpha}}} (x - wt^{\frac{1}{\alpha+1}})^{-\alpha} F'(w)dw = t^{-\frac{\alpha}{\alpha+1}}\frac{\partial}{\partial x}\int_0^{xt^{-\frac{1}{\alpha+1}}} (xt^{-\frac{1}{\alpha+1}} - w)^{-\alpha} F'(w)dw$$

$$= \Gamma(1-\alpha)t^{-1}\frac{\partial}{\partial \xi} D^\alpha F(\xi). \tag{4.78}$$

Hence, if u satisfies $(4.76)_1$, recalling the identity (3.1) we obtain that

$$-\frac{1}{1+\alpha}\xi F'(\xi) - \partial^\alpha F'(\xi) = 0.$$

We will proceed as follows. At first we will solve the auxiliary problem for function F with boundary conditions $F(0) = c_1$, $I^{1-\alpha}F'(0) = c_2$ on the interval $[0,R]$, where $R > 0$, $c_2 < 0$ are arbitrary constants and c_1 comes from $(4.76)_2$. Then, we will propose the formula for the family $\{s\}_R$ and we will choose the constant $c_2 = c_2(R)$ such that the pair $u^R(x,t) = F^R(xt^{-\frac{1}{1+\alpha}})$ and s^R is a solution to $(4.76)_1$, $(4.76)_3$. Then we will choose $R = c_0 > 0$ such that $F(c_0) = 0$, which will guarantee that the pair (u^{c_0}, s^{c_0}) satisfies the whole system (4.76).

Lemma 4.20. *Let us consider the problem*

$$\begin{cases} \partial^\alpha F'(\xi) = -\frac{\xi}{\alpha+1} F'(\xi) & \text{for} \quad 0<\xi<R, \\ F(0) = c_1, \quad I^{1-\alpha} F'(0) = c_2, \end{cases} \tag{4.79}$$

where $c_1 > 0$, $R > 0$, $c_2 < 0$ are fixed constants and $I^{1-\alpha}F'(0) := \lim_{\xi\to 0} I^{1-\alpha}F'(\xi)$. Then, there exists exactly one solution to (4.79) which belongs to

$$X_{R,c_1,c_2} := \{v \in C^1((0,R]) : \xi^{1-\alpha} v' \in C([0,R]),\ v(0) = c_1,\ I^{1-\alpha}v'(0) = c_2\}.$$

Furthermore, the solution is given by the formula

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)}\left[\xi^\alpha + \Gamma(\alpha+1)\xi^\alpha \sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k (i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right], \tag{4.80}$$

where the series is uniformly convergent on $[0,R]$. Finally, if we define

$$u(x,t) := F(xt^{-\frac{1}{1+\alpha}}), \tag{4.81}$$

then $u(0,t) = c_1$ and u satisfies $(4.76)_1$ on $\{(x,t) : 0 < x < Rt^{\frac{1}{\alpha+1}}, 0 < t < \infty\}$.

Proof. At first we will rewrite (4.79) in the integral form. Let us assume that F belonging to X_{R,c_1,c_2} satisfies (4.79). We apply I^α to both sides of $(4.79)_1$. Since $F' \in L^1(0,R)$ from identity (4.79) we obtain that also $\partial^\alpha F' \in L^1(0,R)$. Hence, we may apply Proposition 2.29 to obtain

$$F'(\xi) = c_2 \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\alpha+1} I^\alpha(\xi F')(\xi). \tag{4.82}$$

Integrating this identity and applying Proposition 2.22 we arrive at

$$F(\xi) = c_1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} c_2 - \frac{1}{\alpha+1} I^\alpha I(\xi F')(\xi).$$

We note that

$$\int_0^\xi pF'(p)dp = \xi F(\xi) - \int_0^\xi F(p)dp, \quad \text{i.e.} \quad I(\xi F') = \xi F - IF.$$

Denoting by E the identity operator, we get

$$F(\xi) = c_1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} c_2 + \frac{1}{\alpha+1} I^\alpha (I - \xi E) F(\xi).$$

The above identity may be written in the following form

$$F(\xi) = G(\xi) + KF(\xi), \tag{4.83}$$

where

$$G(\xi) = c_1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} c_2, \quad KF(\xi) = \frac{1}{\alpha+1} I^\alpha (I - \xi E) F(\xi).$$

Let us find a solution to (4.83). Applying the operator K to both sides of (4.83) we obtain

$$KF(\xi) = KG(\xi) + K^2 F(\xi).$$

Iterating this procedure, we arrive at

$$F(\xi) = \sum_{k=0}^{n} K^k G(\xi) + K^{n+1} F(\xi) \quad \text{for any} \quad n \in \mathbb{N}. \tag{4.84}$$

We note that if F belongs to $C([0,R])$, then $K^n F \to 0$ uniformly on $[0,R]$. Indeed, making use of Example 2.1, we may calculate that

$$I^\alpha (I+\xi E)\xi^\beta = \frac{\Gamma(\beta+3)}{\Gamma(\beta+\alpha+2)(\beta+1)}\xi^{\beta+\alpha+1}.$$

Hence, we have

$$|K^n F(\xi)| \le \|F\|_{C([0,R])}\frac{1}{(\alpha+1)^n}(I^\alpha(I+\xi E))^n 1 = \|F\|_{C([0,R])}\frac{\xi^{n(\alpha+1)}\prod_{k=0}^{n-1}(k(\alpha+1)+3)}{(1+\alpha)^n\Gamma(n(\alpha+1)+1)}$$
$$= \|F\|_{C([0,R])}\frac{\xi^{n(\alpha+1)}\prod_{k=0}^{n-1}(k+\frac{3}{\alpha+1})}{\Gamma(n(\alpha+1)+1)} \le 2\|F\|_{C([0,R])} R^{n(\alpha+1)}\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha n)} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus, we may pass to the limit in (4.84) to obtain that

$$F(\xi) = \sum_{k=0}^{\infty} K^k G(\xi). \tag{4.85}$$

We will calculate the sum of the series and we will show that it is uniformly convergent on $[0,R]$. At first, we note that for any $n \in \mathbb{N}\setminus\{0\}$ we have $(I-\xi E)^n 1 = 0$, thus

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{1}{(1+\alpha)^k}[I^\alpha(I-\xi E)]^k\xi^\alpha.$$

Furthermore, from Example 2.1 we may infer that

$$I^\alpha(I-\xi E)\xi^\beta = -\frac{\beta\Gamma(\beta+1)}{\Gamma(\beta+\alpha+2)}\xi^{\beta+\alpha+1}. \tag{4.86}$$

We will show by induction that for every $k\in\mathbb{N}$, $k\ge 1$ we have

$$\frac{1}{\Gamma(1+\alpha)}[I^\alpha(I-\xi E)]^k\xi^\alpha = (-\xi^{1+\alpha})^k\xi^\alpha\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}. \tag{4.87}$$

For $k=1$, applying (4.86) with $\beta=\alpha$ we arrive at

$$\frac{1}{\Gamma(1+\alpha)}I^\alpha(I-\xi E)\xi^\alpha = -\xi^{2\alpha+1}\frac{\alpha}{\Gamma(2\alpha+2)},$$

which is equal to (4.87) for $k=1$. Let us assume that for a fixed $k\ge 1$ identity (4.87) is satisfied. Then

$$\frac{1}{\Gamma(1+\alpha)}[I^\alpha(I-\xi E)]^{k+1}\xi^\alpha = \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}I^\alpha(I-\xi E)[(-\xi^{1+\alpha})^k\xi^\alpha].$$

Making use of (4.86) with $\beta=(1+\alpha)k+\alpha$ we get that

$$\frac{1}{\Gamma(1+\alpha)}[I^\alpha(I-\xi E)]^{k+1}\xi^\alpha = \prod_{i=1}^{k}(i\alpha+i-1)\cdot\frac{(-1)^{k+1}[(1+\alpha)k+\alpha]}{\Gamma((1+\alpha)k+2\alpha+2)}\xi^{(1+\alpha)k+2\alpha+1}$$
$$= (-\xi^{1+\alpha})^{k+1}\xi^\alpha\frac{\prod_{i=1}^{k+1}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+2))}.$$

Hence, by the principle of mathematical induction we obtain (4.87). From identity (4.87) follows that function F defined by (4.85) is given by the formula

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)}\left[\xi^\alpha + \Gamma(\alpha+1)\xi^\alpha\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right].$$

We will show that the series above is uniformly absolutely convergent. Indeed, let us denote

$$a_k = \frac{R^{(1+\alpha)k+\alpha}}{(1+\alpha)^k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}.$$

Then, we may calculate

$$\frac{a_{k+1}}{a_k} = R^{\alpha+1}\frac{k(\alpha+1)+\alpha}{1+\alpha}\frac{\Gamma((\alpha+1)k+\alpha+1)}{\Gamma((\alpha+1)k+2(\alpha+1))}$$

$$\leq \frac{R^{\alpha+1}}{\alpha+1}\frac{\Gamma((\alpha+1)k+\alpha+2)}{\Gamma((\alpha+1)k+\alpha+2+\alpha)} = \frac{R^{\alpha+1}}{\alpha+1}\frac{B(\alpha,(\alpha+1)k+\alpha+2)}{\Gamma(\alpha)} \longrightarrow 0 \text{ as } k \to \infty.$$

Thus, by the d'Alembert criterion the series in (4.80) is uniformly absolutely convergent. Now we will check that F defined by (4.80) actually satisfies (4.83). Let us calculate KF. We note that

$$\frac{1}{\alpha+1}I^\alpha(I-\xi E)c_1 = 0,$$

hence

$$KF(\xi) = \frac{1}{\alpha+1}I^\alpha(I-\xi E)\left[\frac{c_2}{\Gamma(\alpha+1)}\left[\xi^\alpha + \Gamma(\alpha+1)\xi^\alpha\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right]\right].$$

Integrating the series term by term and making use of identity (4.86) we have

$$\frac{1}{\alpha+1}I^\alpha(I-\xi E)F(\xi) = -\frac{\alpha}{\alpha+1}c_2\frac{\xi^{2\alpha+1}}{\Gamma(2(\alpha+1))}$$

$$-\frac{1}{(\alpha+1)}c_2\sum_{k=1}^{\infty}\left(\frac{-1}{1+\alpha}\right)^k\frac{\xi^{(1+\alpha)k+2\alpha+1}}{\Gamma((1+\alpha)(k+2))}[(1+\alpha)k+\alpha]\prod_{i=1}^{k}(i\alpha+i-1)$$

$$= -\frac{\alpha}{\alpha+1}c_2\frac{\xi^{2\alpha+1}}{\Gamma(2(\alpha+1))} + c_2\xi^\alpha\sum_{k=2}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((1+\alpha)(k+1))}$$

$$= c_2\xi^\alpha\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((1+\alpha)(k+1))}.$$

Hence, we verified that function F given by (4.80) satisfies (4.83). Furthermore, the solution to (4.83) belongs to X_{R,c_1,c_2}. Indeed, F given by (4.80) is continuous as a uniform limit of continuous functions. By identity (4.83), we obtain that $F(0) = c_1$ and

$$(I^{1-\alpha}F')(0) = c_2 + \frac{1}{1+\alpha}(D^\alpha I^\alpha(I-\xi E)F)(0) = c_2 + \frac{1}{1+\alpha}((I-\xi E)F)(0) = c_2.$$

In order to show $\xi^{1-\alpha}F' \in C([0,R])$ we differentiate the series in (4.80) term by term.

$$\frac{d}{d\xi}\left[\xi^\alpha\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right] = \xi^{\alpha-1}\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1)+1)}. \tag{4.88}$$

We will show that this series is absolutely convergent uniformly for $\xi \in [0,R]$. Let us denote

$$b_k = \left(\frac{R^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1)+1)}.$$

Then, we may calculate

$$\frac{b_{k+1}}{b_k} = \frac{R^{1+\alpha}}{1+\alpha}\cdot\frac{[(k+1)(\alpha+1)-1]\Gamma((\alpha+1)(k+1)+1)}{\Gamma((\alpha+1)(k+2)+1)}$$

$$\leq \frac{R^{1+\alpha}}{1+\alpha}\cdot\frac{\Gamma((\alpha+1)(k+1)+2)}{\Gamma((\alpha+1)(k+2)+1)} = \frac{R^{1+\alpha}B(\alpha,(\alpha+1)(k+1)+2)}{(1+\alpha)\Gamma(\alpha)} \longrightarrow 0, \text{ as } k \to \infty.$$

Hence, the series in (4.88) is uniformly absolutely convergent, which leads to $\xi^{1-\alpha}F' \in C([0,R])$. Now we will show that F satisfies (4.79). Since $F' \in L^1(0,R)$ we may apply D^α to (4.83) to obtain

$$D^\alpha F(\xi) = c_2 + \frac{1}{1+\alpha} IF(\xi) - \frac{\xi}{1+\alpha}F(\xi),$$

where we made use of Proposition 2.28 and Example 2.1. The right-hand-side is absolutely continuous, hence differentiating the identity above we arrive at

$$\frac{\partial}{\partial x} D^\alpha F(\xi) = -\frac{\xi}{1+\alpha}F'(\xi).$$

The identities (4.77) and (4.78) finish the proof. □

Lemma 4.21. *Let F be a solution to the problem (4.79) given by Lemma 4.20. Then, for every $R > 0$ there holds $F' < 0$ on $(0,R)$. Furthermore, function u defined by (4.81) satisfies $u_t > 0$, $u_x < 0$ on $\{(x,t) : 0 < x < Rt^{\frac{1}{\alpha+1}}, 0 < t < \infty\}$.*

Proof. We note that, since $c_2 < 0$, by (4.80) we have

$$F'(\xi) \to -\infty \quad \text{as} \quad \xi \to 0.$$

Indeed, the derivative of the series in (4.80) vanishes as $\xi \to 0$ and $c_2\xi^{\alpha-1} \to -\infty$ as $\xi \to 0$. Hence, F is decreasing in the neighborhood of zero. We note that F satisfies the assumptions of Lemma 4.11 because by Lemma 4.20 function F is absolutely continuous and F is smooth away from the origin. Let us assume that F admits a local minimum at point $\xi_0 > 0$. Then, $F'(\xi_0) = 0$ and, since F is not constant, by Lemma 4.11 we obtain that $(\frac{\partial}{\partial x}D^\alpha F)(\xi_0) < 0$. It leads to a contradiction with (4.79). Thus, $F' < 0$. The final part of the statement follows from the identities (4.77). □

In the next lemma we obtain the family $(u^R, s^R)_{R>0}$ of solutions to $(4.76)_1$ and $(4.76)_3$.

Lemma 4.22. *For every $c_1 > 0$ and every $R > 0$ the functions*

$$s^R(t) = Rt^{\frac{1}{1+\alpha}}, \tag{4.89}$$

$$u^R(x,t) = c_1 + \frac{\tilde{c}_2}{\Gamma(\alpha+1)}\left[x^\alpha t^{-\frac{\alpha}{\alpha+1}} + \Gamma(\alpha+1)x^\alpha t^{-\frac{\alpha}{\alpha+1}} \sum_{k=1}^{\infty}\left(\frac{-x^{1+\alpha}}{(1+\alpha)t}\right)^k \frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right], \tag{4.90}$$

where

$$\tilde{c}_2 = -\frac{R}{(1+\alpha)\left[1 + \sum_{k=1}^\infty \left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}\right]} \tag{4.91}$$

satisfy the equation $(4.76)_3$. Moreover, u^R is a solution to $(4.76)_1$ with $s(t) = s^R(t)$ and $u^R(0,t) = c_1$.

Proof. We note that $u^R(x,t) = F(xt^{-\frac{1}{1+\alpha}})$ where F is the solution to (4.79) with c_2 equal to $\tilde{c}_2$ whenever $\tilde{c}_2$ given by (4.91) is well defined and negative. It is enough to show that the denominator in the definition of $\tilde{c}_2$ is positive. To this end, let us recall the formula for the function F given by (4.80). Since, by Lemma 4.21, for any $c_2 < 0$ there holds $F' < 0$, we have also $D^\alpha F < 0$. Making use of Example (2.1) we obtain that for any $c_2 < 0$

$$D^\alpha F(R) = c_2 \left[1 + \sum_{k=1}^{\infty} \left(\frac{-R^{1+\alpha}}{1+\alpha} \right)^k \frac{\prod_{i=1}^{k} (i\alpha + i - 1)}{\Gamma((\alpha+1)k+1)} \right].$$

This implies that

$$1 + \sum_{k=1}^{\infty} \left(\frac{-R^{1+\alpha}}{1+\alpha} \right)^k \frac{\prod_{i=1}^{k} (i\alpha + i - 1)}{\Gamma((\alpha+1)k+1)} > 0.$$

Hence, for every $R > 0$ the constant $\tilde{c}_2$ given by (4.91) is well defined and negative. By Lemma 4.20 u^R fulfills $(4.76)_1$ with $s(t) = s^R(t)$ and $u^R(0,t) = c_1$. Moreover,

$$t^{-\frac{\alpha}{\alpha+1}} I^{1-\alpha} F'(\xi) = (I^{1-\alpha} u^R_x)(x,t),$$

hence, $I^{1-\alpha}F'(0) = \tilde{c}_2$ implies $(I^{1-\alpha}u^R_x)(0,t) = \tilde{c}_2 t^{-\frac{\alpha}{\alpha+1}}$. Now we will show that $(u^R, s^R)_{R>0}$ given by (4.89) - (4.91) satisfy $(4.76)_3$. Let us calculate $D^\alpha u^R(x,t)$ for u^R given by (4.90). Applying Example 2.1 we get

$$D^\alpha u^R(x,t) = \tilde{c}_2 t^{-\frac{\alpha}{\alpha+1}} + t^{-\frac{\alpha}{\alpha+1}} \tilde{c}_2 \sum_{k=1}^{\infty} \left(\frac{-x^{1+\alpha}}{(1+\alpha)t} \right)^k \frac{\prod_{i=1}^{k} (i\alpha + i - 1)}{\Gamma((\alpha+1)k+1)}.$$

Hence, for s^R given by (4.89) we have

$$t^{\frac{\alpha}{\alpha+1}} D^\alpha u^R(s^R(t),t) = \tilde{c}_2 + \tilde{c}_2 \sum_{k=1}^{\infty} \left(\frac{-R^{1+\alpha}}{1+\alpha} \right)^k \frac{\prod_{i=1}^{k} (i\alpha + i - 1)}{\Gamma((\alpha+1)k+1)}.$$

Making use of the formula (4.91) we obtain that

$$-D^\alpha u^R(s^R(t),t) = t^{-\frac{\alpha}{\alpha+1}} \frac{R}{1+\alpha} = \dot{s}^R(t)$$

and hence, the functions s^R and u^R defined by (4.89) and (4.90) satisfy $(4.76)_3$ which finishes the proof. □

It remains to choose $R > 0$ such that the pair (u^R, s^R) given by Lemma (4.22) satisfies $u^R(s^R(t),t) = 0$.

Theorem 4.23. *For every $c_1 > 0$ there exists $c_0 > 0$ such that the pair $(u,s) := (u^{c_0}, s^{c_0})$, where (u^{c_0}, s^{c_0}) come from Lemma 4.22 with $R = c_0$, satisfies the system (4.76). Furthermore,*

$$\forall x > 0 \;\; u(x,\cdot), u_t(x,\cdot), u_x(x,\cdot) \in C([s^{-1}(x), \infty)) \tag{4.92}$$

$$\forall t > 0 \;\; u(\cdot,t), u_t(\cdot,t) \in C([0,s(t)]), \;\; u_x(\cdot,t) \in C((0,s(t)]) \tag{4.93}$$

and

$$\forall t > 0 \;\; \frac{\partial}{\partial x} D^\alpha u(\cdot,t) \in C([0,s(t)]). \tag{4.94}$$

Finally, $u > 0$, $u_t > 0$, $u_x < 0$ on $\{(x,t): \;\; 0 < x < s(t), \;\; 0 < t < \infty\}$.

Proof. Let us show that there exists $c_0 > 0$ such that the pair (u^R, s^R) given by Lemma 4.22 with $R = c_0$ satisfies $u^R(s^R(t), t) = 0$. For $\xi = xt^{-\frac{1}{1+\alpha}}$ function u^R defined in (4.90) is given by

$$u^R(x,t) = F(\xi) = c_1 + \tilde{c}_2 g(\xi),$$

where

$$g(\xi) = \left[\frac{\xi^\alpha}{\Gamma(\alpha+1)} + \xi^\alpha \sum_{k=1}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k (i\alpha + i - 1)}{\Gamma((\alpha+1)(k+1))}\right].$$

We note that $g(0) = 0$ and since $\tilde{c}_2 < 0$, from Lemma 4.21 we infer that g is increasing. Applying Lemma 4.11 we obtain that $\frac{\partial}{\partial x} D^\alpha g \leq 0$. Recalling that $\tilde{c}_2$ is given by (4.91) we arrive at

$$F(\xi) = c_1 - \frac{Rg(\xi)}{(\alpha+1)D^\alpha g(R)}.$$

We would like to find $R > 0$ such that $F(R) = 0$. We note that

$$F(R) = c_1 - \frac{Rg(R)}{(\alpha+1)D^\alpha g(R)}.$$

Since the denominator is positive it is enough to show that there exists a positive zero of the function

$$h(R) := c_1(\alpha+1)D^\alpha g(R) - Rg(R).$$

We note that since $D^\alpha g(0) = 1$ we have $h(0) = c_1(\alpha+1) > 0$. On the other hand, since g is absolutely continuous and $g(0) = 0$, we may write $g(R) = I^\alpha D^\alpha g(R)$. Applying $\frac{\partial}{\partial x} D^\alpha g \leq 0$ we may estimate as follows

$$I^\alpha D^\alpha g(R) = \frac{1}{\Gamma(\alpha)} \int_0^R (R-p)^{\alpha-1} D^\alpha g(p) dp \geq \frac{D^\alpha g(R)}{\Gamma(\alpha)} \int_0^R (R-p)^{\alpha-1} dp = \frac{D^\alpha g(R) R^\alpha}{\Gamma(\alpha+1)}.$$

Hence,

$$h(R) = c_1(1+\alpha)D^\alpha g(R) - RI^\alpha D^\alpha g(R) \leq c_1(1+\alpha)D^\alpha g(R) - RD^\alpha g(R)\frac{R^\alpha}{\Gamma(\alpha+1)}.$$

Recalling that $D^\alpha g > 0$ we arrive at $h(R) \to -\infty$ as $R \to \infty$. Hence, since h is continuous we may apply the Darboux property to deduce that there exist $c_0 > 0$ such that $h(c_0) = 0$, which implies $F(c_0) = 0$. Moreover, for $s(t) = c_0 t^{\frac{1}{1+\alpha}}$ there holds $u(s(t), t) = u(c_0 t^{\frac{1}{1+\alpha}}, t) = F(c_0) = 0$. The regularity results (4.92) and (4.93) immediately follows from identities (4.77) and regularity of F established in Lemma 4.20. To show (4.94) we note that since F satisfies (4.79), the continuity of $\xi F(\xi)$ implies $\frac{\partial}{\partial x} D^\alpha F \in C([0, R])$. This together with identity (4.78) leads to (4.94). □

Corollary 4.24. *The solution (u, s) obtained in Theorem 4.23 satisfies the regularity assumptions (2.26), (2.27) which were necessary to derive the model.*

Chapter 5

A special solution to time-fractional Stefan problem

In this chapter we will find a special solution to the model derived in Theorem 2.37. We note that the results presented in this chapter come from [14]. We will look for a self-similar solution to the time-fractional Stefan problem in the domain

$$U = \{(x,t) \in \mathbb{R} \times (0,\infty) : \ \ 0 < x < s(t)\}, \tag{5.1}$$

where $(s(t),t)$ is a free boundary. We impose a constant positive Dirichlet boundary condition on the left boundary and we assume that $s(0) = 0$. In this case, the problem formulated in Theorem 2.37 takes the following form

$$D^{\alpha}_{s^{-1}(x)}u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \text{in} \quad U, \tag{5.2}$$

$$u(s(t),t) = 0 \quad \text{for every} \quad t > 0, \tag{5.3}$$

$$u(0,t) = \gamma \quad \text{for every} \quad t > 0, \tag{5.4}$$

$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)} \lim_{a \nearrow s(t)} \frac{d}{dt}\left[\int_{s^{-1}(a)}^{t} (t-\tau)^{\alpha-1} u_x(a,\tau)d\tau\right] \quad \text{for every} \quad t > 0, \tag{5.5}$$

where $\gamma > 0$ is given. We are going to prove the following result.

Theorem 5.1. *For any $\gamma > 0$ there exists a pair (u,s) which satisfies (5.2)-(5.5). Furthermore, the solution is given by*

$$s(t) = c_1 t^{\frac{\alpha}{2}}, \tag{5.6}$$

$$u(x,t) = \int_{xt^{-\frac{\alpha}{2}}}^{c_1} H(p, xt^{-\frac{\alpha}{2}}) G_{c_1}(p)dp \quad \textit{in} \quad U, \tag{5.7}$$

where $c_1 = c_1(\alpha,\gamma) > 0$ and

$$G_{c_1}(y) = \frac{1}{\Gamma(1-\alpha)} \int_{y}^{c_1} (1 - c_1^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu \quad \textit{for} \quad 0 \le y \le c_1, \tag{5.8}$$

$$H(p,x) = 1 + \int_{x}^{p} N(p,y)dy \quad \textit{for} \quad 0 \le x \le p, \tag{5.9}$$

$$N(p,y) = \sum_{n=1}^{\infty} M_n(p,y) \quad for \quad 0 \le y \le p, \tag{5.10}$$

where

$$M_1(p,y) = \frac{1}{\Gamma(1-\alpha)} \int_y^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu \quad for \quad 0 \le y \le p \tag{5.11}$$

and

$$M_n(p,y) = \int_y^p M_1(a,y) M_{n-1}(p,a) da \ for \quad 0 \le y \le p \ and \quad n \ge 2. \tag{5.12}$$

For every $R > 0$ the series (5.10) converges uniformly on $W_R = \{(p,y) : \ 0 \le y \le p \le R\}$. Functions M_n, N are positive on $\{(p,y) : \ 0 \le y < p\}$, hence u is positive in U.
For every $a, \lambda > 0$ function u satisfies the scaling property

$$u(x,t) = u(\lambda^a x, \lambda^{\frac{2a}{\alpha}} t) \tag{5.13}$$

and

$$u_x(s(t),t) = 0. \tag{5.14}$$

Furthermore, for every $t > 0$ there holds $u(\cdot,t) \in W^{2,1}(0,s(t))$ and $u_t(x,\cdot) \in C([s^{-1}(x),\infty))$ for every $x > 0$. Finally, we have $u_x(x,\cdot) \in L^\infty(s^{-1}(x),\infty) \cap AC_{loc}([s^{-1}(x),\infty))$ for every $x > 0$ and $u_t(\cdot,t) \in L^1(0,s(t))$, $D^\alpha_{s^{-1}(\cdot)} u(\cdot,t) \in L^1(0,s(t))$ for every $t > 0$. In particular, the pair (u,s) satisfies the assumptions (A1) - (A3).

Corollary 5.2. *If c_1 is a positive constant and*

$$\gamma = \int_0^{c_1} H(p,0) G_{c_1}(p) dp,$$

then (5.6)-(5.7) define a solution to (5.2)-(5.5).

Remark 5.1. *If we replace the Dirichlet condition (5.4) by the Neumann condition*

$$u_x(0,t) = -\beta t^{-\frac{\alpha}{2}}, \ \ \beta > 0,$$

then Theorem 5.1 holds with $c_1 = c_1(\alpha,\beta) > 0$.

The proof will be divided into a few steps and after a proof of Theorem 5.1 we will justify the Remark 5.1. At first we will proceed with formal calculations that will lead us to an appropriate scaling. We introduce parameters $a, b, c, \lambda > 0$ and we define the function

$$u^\lambda(x,t) = \lambda^c u(\lambda^a x, \lambda^b t). \tag{5.15}$$

Our aim is to find a, b, c and the curve $(s(t),t)$ such that, if (u,s) is a solution to (5.2), then $u^\lambda \equiv u$.
At first, we perform calculations. We note that $u_{xx}(x,t) = \lambda^{-c}\lambda^{-2a} u^\lambda_{xx}(\lambda^{-a}x, \lambda^{-b}t)$ and

$$\Gamma(1-\alpha) D^\alpha_{s^{-1}(x)} u(x,t) = \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau = \lambda^{-c}\lambda^{-b} \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} u^\lambda_t(\lambda^{-a}x, \lambda^{-b}\tau) d\tau$$

$$= \lambda^{-c} \int_{\lambda^{-b}s^{-1}(x)}^{t\lambda^{-b}} (t - \lambda^b p)^{-\alpha} u^\lambda_t(\lambda^{-a}x, p) dp = \lambda^{-c}\lambda^{-b\alpha} \int_{\lambda^{-b}s^{-1}(x)}^{t\lambda^{-b}} (t\lambda^{-b} - p)^{-\alpha} u^\lambda_t(\lambda^{-a}x, p) dp$$

$$= \lambda^{-c}\lambda^{-b\alpha}\Gamma(1-\alpha) D^{\alpha}_{\lambda^{-b}s^{-1}(x)} u^{\lambda}(\lambda^{-a}x, \lambda^{-b}t),$$

i.e.

$$D^{\alpha}_{s^{-1}(\lambda^a x)} u(\lambda^a x, \lambda^b x) = \lambda^{-c}\lambda^{-b\alpha} D^{\alpha}_{\lambda^{-b}s^{-1}(\lambda^a x)} u^{\lambda}(x,t).$$

Hence, if the pair (u, s) is a solution to (5.2), then

$$0 = D^{\alpha}_{s^{-1}(\lambda^a x)} u(\lambda^a x, \lambda^b t) - u_{xx}(\lambda^a x, \lambda^b t) + \frac{1}{\Gamma(1-\alpha)}(\lambda^b t - s^{-1}(\lambda^a x))^{-\alpha}$$

$$= \lambda^{-c}\lambda^{-b\alpha} D^{\alpha}_{\lambda^{-b}s^{-1}(\lambda^a x)} u^{\lambda}(x,t) - \lambda^{-c}\lambda^{-2a} u^{\lambda}_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}\lambda^{-b\alpha}(t - \lambda^{-b}s^{-1}(\lambda^a x))^{-\alpha}.$$

Thus, if we set $c = 0$ and

$$b = \frac{2a}{\alpha}, \tag{5.16}$$

then we get

$$0 = D^{\alpha}_{\lambda^{-b}s^{-1}(\lambda^a x)} u^{\lambda}(x,t) - u^{\lambda}_{xx}(x,t) + \frac{1}{\Gamma(1-\alpha)}(t - \lambda^{-b}s^{-1}(\lambda^a x))^{-\alpha}.$$

We observe that, if $s(t)$ satisfies

$$s^{-1}(x) = \lambda^{-b}s^{-1}(\lambda^a x), \tag{5.17}$$

then u and u^{λ} are the solutions to the same equation. Let us find s which satisfies (5.17) with parameters a, b related by (5.16). We have to solve the following equation $s^{-1}(x) = \lambda^{-\frac{2a}{\alpha}} s^{-1}(\lambda^a x)$. Function s^{-1} satisfies this identity if it fulfills the functional equation $g(\lambda x) = \lambda^{\frac{2}{\alpha}} g(x)$. To solve this equation, it is enough to write

$$\frac{g(x) - g(\lambda x)}{x(1-\lambda)} = \frac{g(x)}{x}\frac{1 - \lambda^{\frac{2}{\alpha}}}{1 - \lambda}$$

and take the limit $\lambda \to 1$. Then we get that $g' = \frac{2}{\alpha}\frac{g}{x}$, i.e. $g(x) = cx^{\frac{2}{\alpha}}$. Thus, we obtained that, if there exists a self-similar solution, then the interface may have a form

$$s(t) = c_1 t^{\frac{\alpha}{2}} \tag{5.18}$$

for some positive c_1. If we denote

$$c_0 = c_1^{-\frac{2}{\alpha}}, \tag{5.19}$$

then we may write

$$s^{-1}(x) = c_0 x^{\frac{2}{\alpha}}. \tag{5.20}$$

Our aim is to find a special solution u to the system (5.2), (5.3), (5.5), with function s given by (5.18). We will proceed as follows. At first, we will rewrite the equations (5.2), (5.5) in terms of a new self-similarity variable. Subsequently, we will show that if u is appropriately regular, self-similar function, then condition (5.5) implies $u_x(s(t), t) = 0$. Then, we will solve the auxiliary problem

$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \text{in} \quad U, \tag{5.21}$$

$$u(s(t),t) = 0, \;\; u_x(s(t),t) = 0 \quad \text{for} \quad t > 0,$$

with s given by (5.18). The next step is to prove that the obtained solution satisfies (5.5). In the final section, we will prove that the solution is positive and that for every $\gamma > 0$ we may find $c_1 > 0$ such that obtained solution satisfies Dirichlet boundary condition $u(0,t) = \gamma$.

5.1. Similarity variable

Let us begin with introducing a similarity variable

$$\xi = tx^{-\frac{2}{\alpha}}. \tag{5.22}$$

We define function f as follows

$$f(\xi) = f(tx^{-\frac{2}{\alpha}}) := u(x,t). \tag{5.23}$$

In the next proposition we establish how the expected regularity properties of u transforms to the properties of f. Furthermore, we rewrite the conditions (5.2), (5.3), (5.5) in terms of f and prove that (5.5) implies vanishing of derivative of f at point c_0.

Proposition 5.3. *Let us assume that s is given by (5.18) with any fixed $c_1 > 0$ and u is a self-similar solution to (5.2), (5.3), (5.5), where the similarity variable is given by (5.22). Suppose that u has following regularity. For $k > 1$ and every $t > 0$ there hold $u_x(\cdot,t) \in L^1(0,s(t))$, $u_{xx}(\cdot,t) \in L^1(s(t)/k, s(t))$. Then, the function f defined by (5.23) satisfies $f' \in L^1(c_0,\infty) \cap AC([c_0, k^{\frac{2}{\alpha}}c_0])$, $f \in C^2(c_0, k^{\frac{2}{\alpha}}c_0)$ and for $\xi \in (c_0, k^{\frac{2}{\alpha}}c_0)$ we have*

$$\frac{1}{\Gamma(1-\alpha)}\int_{c_0}^{\xi}(\xi-p)^{-\alpha}f'(p)dp = \left(\frac{2}{\alpha}\right)^2\xi^2 f''(\xi) + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right]\xi f'(\xi) - \frac{(\xi-c_0)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{5.24}$$

$$f(c_0) = 0, \tag{5.25}$$

$$\left(\frac{\alpha}{2}\right)^2 c_0^{-2}\Gamma(\alpha) = \lim_{b\searrow c_0}\frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right]. \tag{5.26}$$

The identity (5.24) together with regularity of f implies

$$\lim_{\xi\searrow c_0}(\xi-c_0)^{\alpha}f''(\xi) = \left(\frac{\alpha}{2}\right)^2\frac{c_0^{-2}}{\Gamma(1-\alpha)}, \tag{5.27}$$

while from (5.26) we deduce

$$f'(c_0) = 0. \tag{5.28}$$

Proof. Let us begin with a simple calculation,

$$u_t(x,\tau) = f'(\tau x^{-\frac{2}{\alpha}})x^{-\frac{2}{\alpha}}, \tag{5.29}$$

$$u_x(x,t) = -\frac{2}{\alpha}f'(tx^{-\frac{2}{\alpha}})tx^{-\frac{2}{\alpha}-1}, \tag{5.30}$$

$$u_{xx}(x,t)=\left(\frac{2}{\alpha}\right)^2 f''(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})^2x^{-2}+\frac{2}{\alpha}(\frac{2}{\alpha}+1)f'(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})x^{-2}. \tag{5.31}$$

Applying the substitution $p=\tau x^{-\frac{2}{\alpha}}$ we get

$$D^{\alpha}_{s^{-1}(x)}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_{c_0x^{\frac{2}{\alpha}}}^{t}(t-\tau)^{-\alpha}f'(\tau x^{-\frac{2}{\alpha}})x^{-\frac{2}{\alpha}}d\tau$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_{c_0}^{tx^{-\frac{2}{\alpha}}}(t-x^{\frac{2}{\alpha}}p)^{-\alpha}f'(p)dp=x^{-2}\frac{1}{\Gamma(1-\alpha)}\int_{c_0}^{tx^{-\frac{2}{\alpha}}}(tx^{-\frac{2}{\alpha}}-p)^{-\alpha}f'(p)dp.$$

Furthermore, we have

$$(t-c_0x^{\frac{2}{\alpha}})^{-\alpha}=x^{-2}(tx^{-\frac{2}{\alpha}}-c_0)^{-\alpha}.$$

After having inserted these results in equation (5.2) with s given by (5.18), we obtain (5.24). To show that (5.25) holds, it is enough to notice that, since the function u vanishes on the free boundary, we have

$$0=u(s(t),t)=u(c_1t^{\frac{\alpha}{2}},t)=f(c_0),$$

where we used (5.19). Now, we will prove the regularity results. By (5.30) we get

$$\infty>\int_0^{s(t)}|u_x(x,t)|dx=\frac{2}{\alpha}\int_0^{s(t)}|f'(tx^{-\frac{2}{\alpha}})|tx^{-\frac{2}{\alpha}-1}dx=\int_{c_0}^{\infty}|f'(\xi)|d\xi. \tag{5.32}$$

From (5.31) we obtain in the similar way that

$$\infty>\int_{s(t)/k}^{s(t)}|u_{xx}(x,t)|dx=\int_{s(t)/k}^{s(t)}\left|\left(\frac{2}{\alpha}\right)^2 f''(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})^2x^{-2}+\frac{2}{\alpha}(\frac{2}{\alpha}+1)f'(tx^{-\frac{2}{\alpha}})(tx^{-\frac{2}{\alpha}})x^{-2}\right|dx$$

$$=\int_{c_0}^{k^{\frac{2}{\alpha}}c_0}\left|\frac{2}{\alpha}f''(\xi)\xi^{1+\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}+(\frac{2}{\alpha}+1)f'(\xi)\xi^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}\right|d\xi$$

$$\geq\frac{2}{\alpha}c_0^{1+\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}\int_{c_0}^{k^{\frac{2}{\alpha}}c_0}|f''(\xi)|\,d\xi-(\frac{2}{\alpha}+1)kc_0^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}\int_{c_0}^{\infty}|f'(\xi)|d\xi \text{ for every } t>0$$

and as a consequence we obtain

$$\int_{c_0}^{k^{\frac{2}{\alpha}}c_0}|f''(\xi)|\,d\xi<\infty. \tag{5.33}$$

The estimates (5.32) and (5.33) lead to $f'\in AC([c_0,k^{\frac{2}{\alpha}}c_0])$. Making use of the absolute continuity of f' in identity (5.24) we deduce that $f\in C^2(c_0,k^{\frac{2}{\alpha}}c_0)$. Hence, we obtained postulated regularity results. Now, we shall rewrite the condition (5.5) in terms of the function f. We will show that it leads to (5.26). Let us fix $a\in(s(t)/k,s(t))$. Applying the substitution $p=a^{-\frac{2}{\alpha}}\tau$ we get that

$$A\equiv\frac{d}{dt}\left[\int_{s^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right]=-\frac{2}{\alpha}\frac{d}{dt}\left[\int_{c_0a^{\frac{2}{\alpha}}}^{t}(t-\tau)^{\alpha-1}f'(\tau a^{-\frac{2}{\alpha}})\tau a^{-\frac{2}{\alpha}-1}d\tau\right]$$

$$=-\frac{2}{\alpha}a^{\frac{2}{\alpha}-1}\frac{d}{dt}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(t-a^{\frac{2}{\alpha}}p)^{\alpha-1}pf'(p)dp\right]=-\frac{2}{\alpha}a\frac{d}{dt}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}pf'(p)dp\right].$$

The integration by parts formula leads to

$$A\equiv-\frac{2}{\alpha}a\frac{d}{dt}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}\frac{(ta^{-\frac{2}{\alpha}}-p)^{\alpha}}{\alpha}\left(f'(p)+pf''(p)\right)dp+\frac{(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha}}{\alpha}c_0f'(c_0)\right].$$

By the continuity of second derivatives of f in $(c_0, k^{\frac{2}{\alpha}}c_0)$ we have

$$\lim_{p\nearrow ta^{-\frac{2}{\alpha}}} \frac{(ta^{-\frac{2}{\alpha}}-p)^{\alpha}}{\alpha}\left(f'(p)+pf''(p)\right)=0.$$

Therefore, we obtain

$$A=-\frac{2}{\alpha}a^{1-\frac{2}{\alpha}}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}\left(f'(p)+pf''(p)\right)dp+(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha-1}c_0f'(c_0)\right].$$

Since $f'\in AC([c_0,k^{\frac{2}{\alpha}}c_0])$ we get

$$\lim_{a\nearrow s(t)}\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}f'(p)dp=0.$$

Applying these results together with (5.18) in (5.5) we obtain that

$$\frac{\alpha}{2}c_1t^{\frac{\alpha}{2}-1}=\frac{1}{\Gamma(\alpha)}\frac{2}{\alpha}c_1^{1-\frac{2}{\alpha}}t^{\frac{\alpha}{2}-1}\lim_{a\nearrow s(t)}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}pf''(p)dp+(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha-1}c_0f'(c_0)\right]. \tag{5.34}$$

We note that

$$\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}pf''(p)dp$$

$$=-\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha}f''(p)dp+ta^{-\frac{2}{\alpha}}\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}f''(p)dp.$$

Moreover,

$$\lim_{a\nearrow s(t)}\left|\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha}f''(p)dp\right|\le\lim_{a\nearrow s(t)}(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha}\int_{c_0}^{ta^{-\frac{2}{\alpha}}}\left|f''(p)\right|dp=0.$$

Making use of this convergence in (5.34), we obtain

$$\left(\frac{\alpha}{2}\right)^2c_1^{\frac{2}{\alpha}}\Gamma(\alpha)=c_0\lim_{a\nearrow s(t)}\left[\int_{c_0}^{ta^{-\frac{2}{\alpha}}}(ta^{-\frac{2}{\alpha}}-p)^{\alpha-1}f''(p)dp+(ta^{-\frac{2}{\alpha}}-c_0)^{\alpha-1}f'(c_0)\right],$$

i.e.

$$\left(\frac{\alpha}{2}\right)^2c_0^{-2}\Gamma(\alpha)=\lim_{b\searrow c_0}\frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right],$$

where we applied the equality

$$\int_{c_0}^{b}(b-p)^{\alpha-1}f''(p)dp=\frac{d}{db}\left[\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp\right]-(b-c_0)^{\alpha-1}f'(c_0). \tag{5.35}$$

Thus, we arrive at (5.26). To prove (5.27), we notice that from the equation (5.24) we get

$$\left(\frac{2}{\alpha}\right)^2(\xi-c_0)^{\alpha}\xi^2f''(\xi)$$

$$=\frac{(\xi-c_0)^{\alpha}}{\Gamma(1-\alpha)}\int_{c_0}^{\xi}(\xi-p)^{-\alpha}f'(p)dp-\left[\left(\frac{2}{\alpha}\right)^2+\frac{2}{\alpha}\right](\xi-c_0)^{\alpha}\xi f'(\xi)+\frac{1}{\Gamma(1-\alpha)}.$$

The function f' is absolutely continuous on a neighborhood of c_0 thus, taking the limit at $\xi=c_0$ we obtain (5.27).

It remains to show that (5.26) implies $f'(c_0)=0$. We note that

$$\frac{d}{db}\int_{c_0}^{b}(b-p)^{\alpha-1}f'(p)dp=\Gamma(\alpha)\partial_{c_0}^{1-\alpha}f'(b).$$

We fix $\varepsilon > 0$. Then, from (5.26), there exists $x_0 > c_0$ such that for every $x \in (c_0, x_0)$

$$\left(\frac{\alpha}{2c_0}\right)^2 - \varepsilon \leq \partial_{c_0}^{1-\alpha} f'(x) \leq \left(\frac{\alpha}{2c_0}\right)^2 + \varepsilon.$$

We note that, since f' is absolutely continuous by Proposition 2.29 we have $I_{c_0}^{1-\alpha}\partial_{c_0}^{1-\alpha} f' = f'$. Applying $I_{c_0}^{1-\alpha}$ to the above inequalities and making use of Example 2.2 we obtain that for every $x \in (c_0, x_0)$

$$\left[\left(\frac{\alpha}{2c_0}\right)^2 - \varepsilon\right] \frac{(x-c_0)^{1-\alpha}}{\Gamma(2-\alpha)} \leq f'(x) \leq \left[\left(\frac{\alpha}{2c_0}\right)^2 + \varepsilon\right] \frac{(x-c_0)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

The last pair of inequalities is equivalent with

$$\lim_{x \to c_0} \frac{f'(x)}{(x-c_0)^{1-\alpha}} = \left(\frac{\alpha}{2c_0}\right)^2 \frac{1}{\Gamma(2-\alpha)}$$

and in particular $f'(c_0) = 0$. This way we finished the proof of Proposition 5.3. □

We note that, the converse statement also holds. Reverting the calculations, we obtain the following result.

Corollary 5.4. *Assume that $k > 1$, $c_0 > 0$ and function f is such that $f' \in L^1(c_0, \infty)$, $f' \in AC([c_0, k^{\frac{2}{\alpha}} c_0])$, $f \in C^2(c_0, k^{\frac{2}{\alpha}} c_0)$ and for $\xi \in (c_0, k^{\frac{2}{\alpha}} c_0)$ the equality (5.24) holds. Then $u(x,t) := f(tx^{-\frac{2}{\alpha}})$ satisfies*

$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \text{for} \quad s(t)/k < x < s(t), \ 0 < t,$$

where $s(t)$ is defined in (5.18) with c_1 given by (5.19). Furthermore, for every $t > 0$ there holds $u_x(\cdot, t) \in W^{1,1}(s(t)/k, s(t))$ and for every $x > 0$ there holds $u_t(x, \cdot) \in AC([s^{-1}(x), s^{-1}(kx)])$. If in addition f satisfies (5.25), then $u(s(t),t) = 0$. Moreover, if f satisfies (5.26) then u fulfills (5.5). As a consequence of (5.24) and (5.26), (5.27) and (5.28) hold and then $u_x(s(t),t) = 0$.

5.2. Existence of a self-similar solution

Now, we shall find the solution to the problem (5.24)-(5.26). As it was proven in the previous section, if the solution exists, then it also satisfies (5.28) so, it is convenient to consider the space

$$X_R := \{f \in C^1([c_0, R]) : \ \ f(c_0) = f'(c_0) = 0\},$$

for $R \in (c_0, \infty)$. Firstly, we transform the equation (5.24) into a weaker form and we obtain the existence of the solution to the transformed equation in the space X_R.

Let us apply the integral I_{c_0} to both sides of (5.24). Then, by Proposition 2.22 we have

$$I_{c_0}^{2-\alpha} f'(\xi) = \left(\frac{2}{\alpha}\right)^2 \int_{c_0}^{\xi} \tau^2 f''(\tau) d\tau + \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right] \int_{c_0}^{\xi} \tau f'(\tau) d\tau - \frac{(\xi - c_0)^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{5.36}$$

If we integrate by parts and take into account that $f(c_0) = 0$, $f'(c_0) = 0$, then we obtain

$$\int_{c_0}^{\xi} \tau f'(\tau) d\tau = \xi f(\xi) - \int_{c_0}^{\xi} f(\tau) d\tau$$

and
$$\int_{c_0}^{\xi} \tau^2 f''(\tau)d\tau = \xi^2 f'(\xi) - 2\int_{c_0}^{\xi} \tau f'(\tau)d\tau = \xi^2 f'(\xi) - 2\xi f(\xi) + 2\int_{c_0}^{\xi} f(\tau)d\tau.$$
Inserting these calculations in (5.36) we arrive at
$$I_{c_0}^{1-\alpha} f(\xi) = \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\int_{c_0}^{\xi} f(\tau)d\tau - \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\xi f(\xi) + \left(\frac{2}{\alpha}\right)^2 \xi^2 f'(\xi) - \frac{(\xi - c_0)^{1-\alpha}}{\Gamma(2-\alpha)}.$$
We apply again I_{c_0} to both sides and integrate by parts to get
$$I_{c_0}^{2-\alpha} f(\xi) = \left[\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right] I_{c_0}^2 f(\xi) - \left[3\left(\frac{2}{\alpha}\right)^2 - \frac{2}{\alpha}\right]\int_{c_0}^{\xi} \tau f(\tau)d\tau + \left(\frac{2}{\alpha}\right)^2 \xi^2 f(\xi) - \frac{(\xi - c_0)^{2-\alpha}}{\Gamma(3-\alpha)}.$$
The above equality has the following form
$$f(\xi) = Kf(\xi) + g(\xi), \tag{5.37}$$
where
$$Kf(\xi) = \left(\frac{\alpha}{2}\right)^2 \xi^{-2} I_{c_0}^{2-\alpha} f(\xi) + \left[\frac{\alpha}{2} - 1\right]\xi^{-2} I_{c_0}^2 f(\xi) + \left[3 - \frac{\alpha}{2}\right]\xi^{-2}\int_{c_0}^{\xi} \tau f(\tau)d\tau$$
and
$$g(\xi) = \left(\frac{\alpha}{2}\right)^2 \xi^{-2}\frac{(\xi - c_0)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Proposition 5.5. *Assume that $R \in (c_0, \infty)$. Then there exists a unique $f \in X_R$ solution to (5.37). Furthermore, the obtained solution belongs to $C^2(c_0, R)$ and it satisfies (5.24) on (c_0, R).*

Proof. At first, we note that $g \in X_R$ and the operator K is linear and bounded on X_R. Furthermore, the range of K is contained in $C^2([c_0, R])$, hence, K is compact operator in X_R and by Fredholm alternative the equation (5.37) has a unique solution provided, the homogeneous equation has only one solution. From the estimate
$$|Kf(\xi)| \le \left[\left(\frac{\alpha}{2}\right)^2 c_0^{-2}\frac{(\xi - c_0)^{1-\alpha}}{\Gamma(2-\alpha)} + (1 - \frac{\alpha}{2})c_0^{-2}(\xi - c_0) + (3 - \frac{\alpha}{2})c_0^{-1}\right]\int_{c_0}^{\xi} |f(\tau)|d\tau$$
and Gronwall lemma we deduce that the only solution in X_R of $f - Kf = 0$ is $f \equiv 0$. Hence, there exists exactly one $f \in X_R$ which satisfies (5.37). Since the right hand side of (5.37) belongs to $C^2((c_0, R))$, then so does f. Hence, we may invert the calculations leading to identity (5.37) and we obtain that f satisfies (5.24) on (c_0, R). □

Proposition 5.6. *For every $0 < c_0 < R < \infty$ there exists exactly one f belonging to $C^1([c_0, R]) \cap C^2(c_0, R)$ which satisfies the system (5.24) - (5.28).*

Proof. It remains to show that the solution obtained in Proposition 5.5 satisfies (5.26) and (5.27). We note that (5.27) is a simple consequence of (5.24) and continuity of f'. Let us show (5.26). We fix $\varepsilon > 0$. Then, by (5.27) there exists $\xi_0 > c_0$ such that for every $c_0 < \xi < \xi_0$
$$\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon \le (\xi - c_0)^{\alpha} f''(\xi) \le \left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon.$$

Hence, for every $c_0 < \xi < \xi_0$
$$\left(\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon\right)(\xi - c_0)^{-\alpha} \leq f''(\xi) \leq \left(\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon\right)(\xi - c_0)^{-\alpha}.$$
Applying $\frac{1}{\Gamma(1-\alpha)} I^{\alpha}_{c_0}$ to both these inequalities and using Example 2.1 we obtain that for every $c_0 < \xi < \xi_0$
$$\left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} - \varepsilon \leq \frac{1}{\Gamma(1-\alpha)} I^{\alpha}_{c_0} f''(\xi) \leq \left(\frac{\alpha}{2}\right)^2 \frac{c_0^{-2}}{\Gamma(1-\alpha)} + \varepsilon.$$
Hence,
$$I^{\alpha}_{c_0} f''(\xi) \to \left(\frac{\alpha}{2}\right)^2 c_0^{-2} \text{ as } \xi \to c_0.$$
If we recall that $f'(c_0) = 0$, then from (5.35) we have
$$\lim_{\xi \to c_0} \frac{d}{d\xi} \int_{c_0}^{\xi} (\xi - p)^{\alpha-1} f'(p) dp = \lim_{\xi \to c_0} \Gamma(\alpha) I^{\alpha}_{c_0} f''(\xi) = \Gamma(\alpha) \left(\frac{\alpha}{2}\right)^2 c_0^{-2}$$
and we arrive at (5.26). □

From Corollary 5.4 and Proposition 5.6 we deduce the following result.

Corollary 5.7. *Let f be the solution to (5.24)-(5.28) given by Proposition 5.6. Then, for every $k \in (1, \infty)$ function $u(x,t) := f(tx^{-\frac{2}{\alpha}})$ satisfies*
$$D^{\alpha}_{s^{-1}(x)} u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t - s^{-1}(x))^{-\alpha} \quad \text{for} \quad s(t)/k < x < s(t), \ \ t > 0,$$
$$u(s(t), t) = 0 \quad \text{for every} \quad t > 0,$$
$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)} \lim_{a \nearrow s(t)} \frac{d}{dt} \left[\int_{s^{-1}(a)}^{t} (t-\tau)^{\alpha-1} u_x(a,\tau) d\tau\right] \quad \text{for every} \quad t > 0,$$
$$u_x(s(t), t) = 0 \quad \text{for every} \quad t > 0,$$
where $s(t)$ is defined by (5.18) with c_1 given by (5.19). Furthermore, for every $t > 0$ there hold $u_x(\cdot, t) \in W^{1,1}(s(t)/k, s(t))$ and $u_t(x, \cdot) \in AC([s^{-1}(x), s^{-1}(kx)])$ for every $x > 0$.

Now, we shall examine the positivity of u given in the above corollary. In the next section we shall show that $f(\xi) > 0$ for each $\xi > c_0$ and we determine the limit of f at infinity.

5.3. Positivity of solution

Proposition 5.8. *The function f given in Proposition 5.6 is positive on (c_0, ∞). Furthermore,*
$$f(\xi) = \int_{\xi^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n G(y)) dy, \tag{5.38}$$
where the constants c_0 and c_1 are related by the formula (5.19) and
$$(Lh)(x) := \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_{\mu}^{c_1} (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} h(p) dp d\mu, \tag{5.39}$$

$$G(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} (1 - c_0 \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{5.40}$$

The series converges uniformly on $[0, c_1]$. *Moreover, if* $F(\mu) := f(\mu^{-\frac{2}{\alpha}})$, *then* $F \in C^1([0, c_1])$ *and* $F'' \in L^1(0, c_1)$.

Proof. In order to prove the positivity of f on (c_0, ∞) we have to transform the equation (5.24). We introduce $\mu := \xi^{-\frac{\alpha}{2}}$ and

$$F(\mu) := f(\mu^{-\frac{2}{\alpha}}) = f(\xi). \tag{5.41}$$

We note that if $\xi \in (c_0, \infty)$, then $\mu \in (0, c_1)$ and $f(c_0) = f'(c_0) = 0$ implies $F(c_1) = F'(c_1) = 0$. We will rewrite the identity (5.24) in terms of function F. We note that

$$F'(\mu) = -\frac{2}{\alpha} \mu^{-\frac{2}{\alpha}-1} f'(\mu^{-\frac{2}{\alpha}}) \tag{5.42}$$

and

$$F''(\mu) = \frac{2}{\alpha}(\frac{2}{\alpha}+1)\mu^{-\frac{2}{\alpha}-2} f'(\mu^{-\frac{2}{\alpha}}) + (\frac{2}{\alpha})^2 \mu^{-\frac{2}{\alpha}-1} \mu^{-\frac{2}{\alpha}-1} f''(\mu^{-\frac{2}{\alpha}}).$$

Hence,

$$\mu^2 F''(\mu) = \left[\left(\frac{2}{\alpha}\right)^2 + \frac{2}{\alpha}\right] \xi f'(\xi) + \left(\frac{2}{\alpha}\right)^2 \xi^2 f''(\xi).$$

Furthermore,

$$\int_\mu^{c_1} (\mu^{-\frac{2}{\alpha}} - p^{-\frac{2}{\alpha}})^{-\alpha} F'(p) dp = -\frac{2}{\alpha} \int_\mu^{c_1} (\mu^{-\frac{2}{\alpha}} - p^{-\frac{2}{\alpha}})^{-\alpha} p^{-\frac{2}{\alpha}-1} f'(p^{-\frac{2}{\alpha}}) dp.$$

Applying the substitution $p^{-\frac{2}{\alpha}} = w$ we get

$$\int_\mu^{c_1} (\mu^{-\frac{2}{\alpha}} - p^{-\frac{2}{\alpha}})^{-\alpha} F'(p) dp = -\int_{c_0}^{\mu^{-\frac{2}{\alpha}}} (\mu^{-\frac{2}{\alpha}} - w)^{-\alpha} f'(w) dw = -\int_{c_0}^{\xi} (\xi - w)^{-\alpha} f'(w) dw.$$

Inserting the result of these calculations in (5.24) we find out that function F satisfies

$$F''(\mu) = -\frac{1}{\Gamma(1-\alpha)} \mu^{-2} \int_\mu^{c_1} (\mu^{-\frac{2}{\alpha}} - p^{-\frac{2}{\alpha}})^{-\alpha} F'(p) dp + \frac{1}{\Gamma(1-\alpha)} \mu^{-2} (\mu^{-\frac{2}{\alpha}} - c_0)^{-\alpha},$$

which is equivalent with

$$F''(\mu) = -\frac{1}{\Gamma(1-\alpha)} \int_\mu^{c_1} (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} F'(p) dp + \frac{1}{\Gamma(1-\alpha)} (1 - c_0 \mu^{\frac{2}{\alpha}})^{-\alpha}. \tag{5.43}$$

Integrating this equality from x to c_1 and recalling that $F'(c_1) = 0$ we get

$$F'(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_\mu^{c_1} (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} F'(p) dp d\mu - \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} (1 - c_0 \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{5.44}$$

We are going to obtain an explicit formula for F and we will show that F is positive in $[0, c_1]$. Since f' is continuous in $[c_0, \infty)$ then (5.42) implies that $F' \in C(0, c_1]$.

Then, identity (5.44) may be written as

$$F'(x) = (LF')(x) - G(x) \tag{5.45}$$

where the operator L and function G are defined by (5.39) and (5.40), respectively. We apply L to both sides of (5.45) and we deduce that

$$F'(x) = (L^2 F')(x) - (G(x) + LG(x)).$$

Iterating this procedure we obtain that for every $n \in \mathbb{N}$ and every $x \in (0, c_1)$ there holds

$$F'(x) = (L^n F')(x) - \sum_{k=0}^{n} (L^k G)(x). \tag{5.46}$$

Let us show that for every fixed $x_0 \in (0, c_1)$

$$\lim_{n\to\infty} \max_{x\in[x_0,c_1]} |(L^n F')(x)| = 0. \tag{5.47}$$

At first we note that for any $x_0 \in [0, c_1]$ and $h \in C([x_0, c_1])$ there holds

$$\|L^n h\|_{C([x_0,c_1])} \leq \|h\|_{C([x_0,c_1])} \|L^n 1\|_{C([x_0,c_1])}. \tag{5.48}$$

Let us focus on the estimate of $L^n 1$. By the Fubini theorem we have

$$\frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_\mu^{c_1} (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} dp d\mu = \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu dp.$$

We note that

$$\int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu \leq \frac{\alpha}{2} B(\frac{\alpha}{2}, 1-\alpha) p, \tag{5.49}$$

where we applied the substitution $w := p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}}$. Hence, we obtain

$$0 < L1(x) \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1 (I_x 1)(c_1) \quad \text{for} \quad x \in [0, c_1). \tag{5.50}$$

We shall show by induction that

$$0 < L^n 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^n (I_x^n 1)(c_1) \quad \text{for} \quad x \in [0, c_1) \tag{5.51}$$

for each $n \in \mathbb{N}$. Indeed, suppose that (5.51) holds for $n = k-1$ and then we have

$$\begin{aligned} L^k 1(x) = LL^{k-1} 1(x) &= \frac{1}{\Gamma(1-\alpha)} \int_x^{c_1} \int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu (L^{k-1} 1)(p) dp \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^{k-1} \int_x^{c_1} \int_x^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu (I_p^{k-1} 1)(c_1) dp \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^{k-1} \int_x^{c_1} \frac{\alpha}{2} B(\frac{\alpha}{2}, 1-\alpha) p (I_p^{k-1} 1)(c_1) dp, \end{aligned}$$

where in the last inequality we used (5.49). Thus, we have

$$L^k 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^k \int_x^{c_1} (I_p^{k-1} 1)(c_1) dp = \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1\right]^k I_x^k 1(c_1)$$

and (5.51) is proven. We note that

$$(I_x^n 1)(c_1) = \frac{1}{\Gamma(n)} \int_x^{c_1} (c_1 - \tau)^{n-1} d\tau = \frac{(c_1 - x)^n}{n!} \tag{5.52}$$

hence, by (5.51) we get

$$0 < L^n 1(x) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2\right]^n \frac{1}{n!}. \tag{5.53}$$

Applying the estimate (5.53) in (5.48) we obtain that

$$\begin{aligned} \max_{x\in[x_0,c_1]} |(L^n F')(x)| &\leq \max_{x\in[x_0,c_1]} |F'(x)| \max_{x\in[x_0,c_1]} |L^n 1(x)| \\ &\leq \max_{x\in[x_0,c_1]} |F'(x)| \left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1^2\right)^n \frac{1}{n!} \end{aligned}$$

and due to the presence of factorial function in the denominator the convergence (5.47) holds. We will show that the series $\sum_{k=0}^{\infty}(L^k G)(x)$ is uniformly convergent on $[0, c_1]$. Indeed, applying the substitution $w := c_0\mu^{\frac{2}{\alpha}}$ in the definition of G we obtain that

$$G(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\alpha}{2}c_1\int_{c_0 x^{\frac{2}{\alpha}}}^{1}(1-w)^{-\alpha}w^{\frac{\alpha}{2}-1}dw.$$

Thus,

$$\max_{x\in[0,c_1]}|G(x)| \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1.$$

Applying estimates (5.48) and (5.53) for $x_0 = 0$ we arrive at

$$\max_{x\in[0,c_1]}|L^n G(x)| \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1^2\right)^n\frac{1}{n!} =: a_n.$$

We note that

$$\frac{a_{n+1}}{a_n} = \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}c_1^2\frac{1}{n+1} \to 0 \text{ as } n\to\infty.$$

Hence, by comparison criterion and d'Alembert criterion for convergence of the series we obtain that $\sum_{k=0}^{\infty}(L^k G)(x)$ is uniformly convergent on $[0, c_1]$. Finally, we may pass to the limit in (5.46) to obtain

$$F'(x) = -\sum_{n=0}^{\infty}(L^n G)(x) \text{ for every } x\in[0,c_1], \tag{5.54}$$

where the right hand side converges uniformly. As a consequence, $F \in C^1([0, c_1])$ and by (5.43) we get $F'' \in L^1(0, c_1)$.

We note that $L^n G(x) > 0$ for $[0, c_1)$ thus,

$$F' < 0 \text{ on } [0, c_1). \tag{5.55}$$

Applying the fundamental theorem of calculus, we may write

$$F(x) = -\int_x^{c_1}F'(y)dy = \int_x^{c_1}\sum_{n=0}^{\infty}(L^n G)(y)dy \quad \text{for every } x\in[0,c_1]. \tag{5.56}$$

Thus, we have obtained that F is positive on $[0, c_1)$. We recall that the functions f and F are related by the equality (5.41) therefore, we proved the claim. □

From Corollary 5.7 and Proposition 5.8 we arrive at the following conclusion.

Corollary 5.9. *Let $c_1 > 0$ and $s(t) = c_1 t^{\frac{\alpha}{2}}$. Let us define*

$$u(x,t) := \int_{xt^{-\frac{\alpha}{2}}}^{c_1}\sum_{n=0}^{\infty}(L^n G(y))dy \quad \textit{for} \quad x\in[0,s(t)], \ t>0,$$

where L and G are given by (5.39) and (5.40), respectively. Then, the above series converges uniformly and for every $n \in \mathbb{N}$ there holds $L^n G(y) > 0$ for every $y \in [0, c_1)$. Moreover, $u(x,t)$ satisfies

$$D^{\alpha}_{s^{-1}(x)}u(x,t) = u_{xx}(x,t) - \frac{1}{\Gamma(1-\alpha)}(t-s^{-1}(x))^{-\alpha} \quad \textit{for} \quad 0<x<s(t),$$

$$u(s(t),t) = 0,$$

$$\dot{s}(t) = -\frac{1}{\Gamma(\alpha)}\lim_{a\nearrow s(t)}\frac{d}{dt}\left[\int_{s^{-1}(a)}^{t}(t-\tau)^{\alpha-1}u_x(a,\tau)d\tau\right],$$

$$u_x(s(t),t) = 0,$$

for every $t > 0$. *Finally, from equality* $u(x,t) = F(xt^{-\frac{\alpha}{2}})$ *we deduce that for every* $t > 0$ $u(\cdot,t) \in W^{2,1}(0,s(t))$ *and for every* $x > 0$ *there holds* $u_t(x,\cdot) \in C([s^{-1}(x),\infty))$.

Corollary 5.10. *Functions* u *and* s *defined in Corollary 5.9 satisfy* $u_x < 0$, $u_t > 0$ *in* $\{(x,t) \in \mathbb{R} \times (0,\infty) : \ 0 < x < s(t)\}$,

$$\forall \ x > 0 \ u_x(x,\cdot) \in L^\infty(s^{-1}(x),\infty) \cap AC_{loc}([s^{-1}(x),\infty)) \tag{5.57}$$

and

$$\forall \ t > 0 \ u_t(\cdot,t) \in L^1(0,s(t)) \ \textit{and} \ D^\alpha_{s^{-1}(\cdot)} u(\cdot,t) \in L^1(0,s(t)). \tag{5.58}$$

In particular, the pair (u,s) *satisfies the assumptions (A1) - (A3).*

Proof. At first, we recall that

$$u_x(x,t) = t^{-\frac{\alpha}{2}} F'(\mu), \quad u_t(x,t) = -\frac{\alpha}{2} x t^{-\frac{\alpha}{2}-1} F'(\mu),$$

where $\mu = xt^{-\frac{\alpha}{2}}$. Hence, by (5.55) we infer $u_x < 0$, $u_t > 0$. Since, $F' \in C([0,c_1])$ and for fixed $x > 0$ μ is continuous and bounded function of t on $[s^{-1}(x),\infty)$, we obtain that $u_x(x,\cdot) \in L^\infty(s^{-1}(x),\infty) \cap C([s^{-1}(x),\infty))$. Let us show that $u_x(x,\cdot)$ is absolutely continuous. We may calculate

$$u_{x,t}(x,t) = -\frac{\alpha}{2} t^{-\frac{\alpha}{2}-1} F'(xt^{-\frac{\alpha}{2}}) - \frac{\alpha}{2} t^{-\alpha} x F''(xt^{-\frac{\alpha}{2}}).$$

Hence, for every $t^* > 0$

$$\int_{s^{-1}(x)}^{t^*} |u_{x,t}(x,t)|\, dt = \int_{c_0 x^{\frac{2}{\alpha}}}^{t^*} \left| -\frac{\alpha}{2} t^{-\frac{\alpha}{2}-1} F'(xt^{-\frac{\alpha}{2}}) - \frac{\alpha}{2} t^{-\alpha} x F''(xt^{-\frac{\alpha}{2}}) \right| dt.$$

Applying the substitution $\mu = xt^{-\frac{\alpha}{2}}$ we have

$$\int_{s^{-1}(x)}^{t^*} |u_{x,t}(x,t)|\, dt \le \int_{xt^{*-\frac{\alpha}{2}}}^{c_1} x^{-1} |F'(\mu)|\, d\mu + \int_{xt^{*-\frac{\alpha}{2}}}^{c_1} x^{\frac{2}{\alpha}-1} \mu^{1-\frac{2}{\alpha}} |F''(\mu)|\, d\mu < \infty,$$

because from Proposition 5.8 we have $F' \in C([0,c_1])$, $F'' \in L^1(0,c_1)$. To prove (5.58), we note that for every $t > 0$

$$\begin{aligned} \|u_t(\cdot,t)\|_{L^1(0,s(t))} &= \int_0^{s(t)} u_t(x,t) dx = -\frac{\alpha}{2} \int_0^{c_1 t^{\frac{\alpha}{2}}} x t^{-\frac{\alpha}{2}-1} F'(xt^{-\frac{\alpha}{2}}) dx \\ &= -\frac{\alpha}{2} t^{\frac{\alpha}{2}-1} \int_0^{c_1} pF'(p)dp < \infty, \end{aligned}$$

because $F' \in C([0,c_1])$. Using this results we obtain further,

$$\int_0^{s(t)} \left| D^\alpha_{s^{-1}(x)} u(x,t) \right| dx = \frac{1}{\Gamma(1-\alpha)} \int_0^{s(t)} \int_{s^{-1}(x)}^t (t-\tau)^{-\alpha} u_t(x,\tau) d\tau dx$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^{s(\tau)} u_t(x,\tau) dx d\tau = -\frac{1}{\Gamma(1-\alpha)} \frac{\alpha}{2} \int_0^{c_1} pF'(p)dp \int_0^t (t-\tau)^{-\alpha} \tau^{\frac{\alpha}{2}-1} d\tau$$

$$= -\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} \int_0^{c_1} pF'(p)dp < \infty.$$

Corollary 5.9 together with (5.57) and (5.58) implies that the pair (u,s) satisfies the assumptions (A1) - (A3). □

5.4. Boundary condition

By Corollary 5.9, for each $c_1 > 0$ we have obtained a self-similar solution to time-fractional Stefan problem $(u, s)_{c_1}$ such that

$$u(0,t) = \int_0^{c_1} \sum_{n=0}^{\infty} (L^n G(y)) dy. \tag{5.59}$$

Now, we address to Dirichlet boundary condition (5.4). We investigate whether for given $\gamma > 0$ it is possible to find $c_1 > 0$ such that $(u, s)_{c_1}$ satisfy (5.2)-(5.5).

For this purpose we write explicitly the dependence of solution on c_1. Recall, that from (5.39), (5.40) and the Fubini theorem we have

$$(L_{c_1} h)(y) = \frac{1}{\Gamma(1-\alpha)} \int_y^{c_1} \int_y^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu h(p) dp, \tag{5.60}$$

$$G_{c_1}(y) = \frac{1}{\Gamma(1-\alpha)} \int_y^{c_1} (1 - c_1^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{5.61}$$

The next proposition provides the representation (5.7) of the self-similar solution.

Proposition 5.11. *If c_1 is positive and $s(t) = c_1 t^{\frac{\alpha}{2}}$, then for $t > 0$ and $x \in [0, s(t)]$ we have*

$$\int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L_{c_1}^n G_{c_1}(y)) dy = \int_{xt^{-\frac{\alpha}{2}}}^{c_1} H(p, xt^{-\frac{\alpha}{2}}) G_{c_1}(p) dp, \tag{5.62}$$

where the function H is defined by (5.9)-(5.12). Furthermore, $H - 1$ is positive on the set $W := \{(p, x) : \ 0 \le x < p\}$ and H is continuous on $\overline{W}$.

Proof. We will find another recursive formula for $L_{c_1}^n G_{c_1}$. For $0 \le y \le p < \infty$ we denote

$$M_1(p, y) := \frac{1}{\Gamma(1-\alpha)} \int_y^p (1 - p^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}})^{-\alpha} d\mu. \tag{5.63}$$

Then, we may write

$$(L_{c_1} h)(y) = \int_y^{c_1} M_1(p, y) h(p) dp.$$

Further, we obtain

$$(L_{c_1}^2 G_{c_1})(y) = \int_y^{c_1} M_1(p, y)(L_{c_1} G_{c_1})(p) dp = \int_y^{c_1} \int_y^r M_1(p, y) M_1(r, p) dp G_{c_1}(r) dr.$$

Thus, if we denote

$$M_2(r, y) := \int_y^r M_1(p, y) M_1(r, p) dp \tag{5.64}$$

then,

$$(L_{c_1}^2 G_{c_1})(y) = \int_y^{c_1} M_2(p, y) G_{c_1}(p) dp.$$

By induction we obtain

$$(L_{c_1}^n G_{c_1})(y) = \int_y^{c_1} M_n(p, y) G_{c_1}(p) dp \quad \text{for} \quad n \ge 1 \tag{5.65}$$

where we set

$$M_n(p, y) := \int_y^p M_1(a, y) M_{n-1}(p, a) da \quad \text{for} \quad n \ge 2. \tag{5.66}$$

Now, we shall obtain the estimate for M_n. By (5.49) we get

$$M_1(p,y) \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p. \tag{5.67}$$

Then,

$$M_2(p,y) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\right]^2 p\int_y^p ada \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^2 (I_y 1)(p).$$

We prove by induction that

$$M_n(p,y) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^n (I_y^{n-1}1)(p), \quad n \geq 2. \tag{5.68}$$

Indeed, if

$$M_k(p,y) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^k (I_y^{k-1}1)(p),$$

then by (5.67) we obtain

$$\begin{aligned} M_{k+1}(p,y) &\leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^k \int_y^p \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} a(I_a^{k-1}1)(p)da \\ &\leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^{k+1} \int_y^p (I_a^{k-1}1)(p)da = \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^{k+1} (I_y^k 1)(p) \end{aligned}$$

hence, we arrive at (5.68). Applying (5.52) in (5.68) we get the following estimate

$$M_n(p,y) \leq \left[\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}p\right]^n \frac{p^{n-1}}{(n-1)!} \quad \text{for} \quad n \geq 2. \tag{5.69}$$

Let us define

$$N(p,y) := \sum_{n=1}^{\infty} M_n(p,y), \;\; 0 \leq y \leq p < \infty. \tag{5.70}$$

If $R > 0$, then by (5.69) the series converges uniformly on the set

$$W_R = \{(p,y) : \;\; 0 \leq y \leq p \leq R\}. \tag{5.71}$$

In particular, N is continuous, non-negative and bounded on W_R for each R positive.

If we sum over n both sides of (5.65), then we get

$$\sum_{n=1}^{\infty} L_{c_1}^n G_{c_1}(y) = \int_y^{c_1} N(p,y)G_{c_1}(p)dp. \tag{5.72}$$

Therefore, we have

$$\int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L_{c_1}^n G_{c_1}(y))dy = \int_{xt^{-\frac{\alpha}{2}}}^{c_1} G_{c_1}(y)dy + \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \int_y^{c_1} N(p,y)G_{c_1}(p)dpdy.$$

If we denote

$$H(p,x) := 1 + \int_x^p N(p,y)dy \quad \text{for} \quad 0 \leq x \leq p, \tag{5.73}$$

then after applying Fubini theorem we obtain (5.62). □

Now, we shall investigate the dependence of the self-similar solution obtained in Corollary 5.9 from the parameter c_1. For this purpose we apply the representation given by Proposition 5.11 and we denote

$$F_{c_1}(x) = \int_x^{c_1} H(p,x)G_{c_1}(p)dp. \tag{5.74}$$

Having in mind that the function H is continuous and bounded, we will examine the continuity of the mapping

$$c_1 \mapsto F_{c_1}(x) = \int_x^{c_1} H(p,x) G_{c_1}(p) dp. \tag{5.75}$$

The precise formulation is stated below.

Proposition 5.12. *Assume that c_1 is positive. Then for every $x \in [0, c_1)$*

$$\lim_{\overline{c}_1 \to c_1} F_{\overline{c}_1}(x) = F_{c_1}(x). \tag{5.76}$$

Moreover, we have

$$\lim_{c_1 \searrow 0} F_{c_1}(0) = 0 \tag{5.77}$$

and

$$\lim_{c_1 \nearrow \infty} F_{c_1}(0) = \infty. \tag{5.78}$$

Furthermore, if $\gamma > 0$, then there exists positive c_1 such that

$$F_{c_1}(0) = \int_0^{c_1} H(p,0) G_{c_1}(p) dp = \gamma. \tag{5.79}$$

Proof. Let us fix $x \in [0, c_1)$ and assume that $\overline{c}_1 > c_1$. Then by formula (5.74) we get

$$F_{\overline{c}_1}(x) - F_{c_1}(x) = \int_{c_1}^{\overline{c}_1} H(p,x) G_{\overline{c}_1}(p) dp + \int_x^{c_1} H(p,x) [G_{\overline{c}_1}(p) - G_{c_1}(p)] dp.$$

We note that H is bounded on $\{(p,x): \; 0 \leq x \leq p \leq \overline{c}_1\}$ and

$$|G_{\overline{c}_1}(p)| \leq \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} \overline{c}_1$$

hence, the first integral converges to zero, if $\overline{c}_1 \searrow c_1$. Next, we recall that after substitution $w := c_1^{-\frac{2}{\alpha}} \mu^{\frac{2}{\alpha}}$ we have

$$G_{c_1}(p) = \frac{\alpha}{2\Gamma(1-\alpha)} c_1 \int_{\overline{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^1 (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw$$

and hence

$$G_{\overline{c}_1}(p) - G_{c_1}(p)$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)} \left[(\overline{c}_1 - c_1) \int_{\overline{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^1 (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw + c_1 \int_{\overline{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^{c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}} (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw \right].$$

The first integral is uniformly bounded by $B(1-\alpha, \frac{\alpha}{2})$ hence, the first term converges to zero, if $\overline{c}_1 \searrow c_1$. The second integral also converges to zero because

$$\int_{\overline{c}_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^{c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}} (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw \leq \sup_{W \subset [0,1], |W| \leq (\frac{\overline{c}_1}{c_1})^{\frac{2}{\alpha}} - 1} \int_W (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw \to 0,$$

if $\overline{c}_1 \searrow c_1$. The case $\overline{c}_1 < c_1$ may be shown similarly. Therefore, we obtained (5.76).

To get (5.77) we note that

$$F_{c_1}(0) = \int_0^{c_1} H(p,0) G_{c_1}(p) dp \leq \|H\|_{L^\infty(W_{c_1})} \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} c_1 \to 0,$$

if $c_1 \searrow 0$.

Recalling that N is non-negative, we may write

$$F_{c_1}(0) \geq \int_0^{c_1} G_{c_1}(p)dp = \frac{\alpha}{2\Gamma(1-\alpha)} c_1 \int_0^{c_1} \int_{c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^1 (1-w)^{-\alpha} w^{\frac{\alpha}{2}-1} dw dp$$

$$\geq \frac{\alpha}{2\Gamma(1-\alpha)} c_1 \int_0^{c_1} \int_{c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}}}^1 (1-w)^{-\alpha} dw dp = \frac{\alpha}{2\Gamma(2-\alpha)} c_1 \int_0^{c_1} (1 - c_1^{-\frac{2}{\alpha}} p^{\frac{2}{\alpha}})^{1-\alpha} dp$$

$$= \frac{\left(\frac{\alpha}{2}\right)^2 c_1^2}{\Gamma(2-\alpha)} B(2-\alpha, \frac{\alpha}{2}) = \frac{\alpha \Gamma(1+\frac{\alpha}{2})}{2\Gamma(2-\frac{\alpha}{2})} c_1^2 \to \infty \text{ as } c_1 \to \infty$$

and we proved (5.78).

Finally, it remains to prove that for each $\gamma \in (0, \infty)$ there exists $c_1 \in (0, \infty)$ such that

$$F_{c_1}(0) = \gamma.$$

From (5.76) we deduce the continuity of $(0, \infty) \ni c_1 \mapsto F_{c_1}(0)$. Applying the Darboux property together with (5.77), (5.78) we deduce that this map is onto $(0, \infty)$. □

To prove Theorem 5.1, it remains to collect the obtained results.

Proof of Theorem 5.1. The result is a direct consequence of Corollary 5.9, Corollary 5.10, Proposition 5.11 and Proposition 5.12. □

Proof of Remark 5.1. We note that Remark 5.1 is a simple consequence of Theorem 5.1. Indeed, from the formula (5.7) we obtain that

$$u_x(0,t) = -t^{-\frac{\alpha}{2}} \left[c_1 \frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} + \int_0^{c_1} N(p,0) G_{c_1}(p) dp \right] =: -t^{-\frac{\alpha}{2}} g(c_1).$$

Since N is continuous and bounded on W_R for every $R > 0$ and G_{c_1} is continuous with respect to c_1, we obtain that g is continuous as well. Furthermore, $g(0) = 0$ and $\lim_{c_1 \to \infty} g(c_1) = \infty$. Thus, Remark 5.1 follows from the Darboux property. □

5.5. Convergence to a solution to the classical Stefan problem

We finish this chapter with a result concerning the convergence of self-similar solutions to the fractional Stefan problem to a solution to the classical Stefan problem. To formulate the result we introduce a new notation. We fix $c_1 > 0$ and for $\alpha \in (0,1)$ we denote by s_α and u_α the solution to fractional Stefan problem (5.2) - (5.5) with $\gamma = \int_0^{c_1} H(p,0) G_{c_1}(p) dp$ given by (5.6) and (5.7). Then we set

$$\widetilde{u}_\alpha(x,t) = \begin{cases} u_\alpha(x,t) & \text{for} \quad t > 0, \ x \in [0, c_1 t^{\frac{\alpha}{2}}] \\ 0 & \text{for} \quad t > 1, \ x \in [c_1 t^{\frac{\alpha}{2}}, c_1 t^{\frac{1}{2}}]. \end{cases} \tag{5.80}$$

Theorem 5.13. *Let us fix $0 < t_* < t^* < \infty$. If $\alpha \nearrow 1$, then $\widetilde{u}_\alpha$ converges uniformly on the set $\{(x,t) : \ t \in [t_*, t^*], \ x \in [0, c_1 t^{\frac{1}{2}}]\}$ to u_1, where u_1 is a solution to the classical Stefan problem corresponding to the free boundary $s_1 := c_1 t^{\frac{1}{2}}$, i.e. s_1 and u_1 satisfy*

$$u_{1,t}(x,t) - u_{1,xx}(x,t) = 0 \quad \textit{for} \quad t > 0, \ x \in (0, s_1(t)), \tag{5.81}$$

$$u_1(s_1(t),t)=0 \quad for \quad t>0, \tag{5.82}$$

$$u_1(0,t)=2ae^{a^2}\int_0^a e^{-w^2}dw \quad for \quad t>0, \quad where\ a=\frac{c_1}{2}, \tag{5.83}$$

$$\frac{d}{dt}s_1(t)=-u_{1,x}(s_1(t),t) \quad for \quad t>0 \tag{5.84}$$

and u_1 is given by the formula

$$u_1(x,t)=2ae^{a^2}\int_{\frac{x}{2\sqrt{t}}}^a e^{-w^2}dw. \tag{5.85}$$

Furthermore, (5.2) converges in the sense of distributions to (5.81).

Proof. Let us fix $c_1>0$. We recall the representation of solutions to the system (5.2) - (5.5) with $\gamma=\int_0^{c_1}H(p,0)G_{c_1}(p)dp$ given in Corollary 5.9:

$$s_\alpha(t)=c_1t^{\frac{\alpha}{2}},$$

$$u_\alpha(x,t)=\int_{xt^{-\frac{\alpha}{2}}}^{c_1}\sum_{n=0}^{\infty}(L^n_{c_1,\alpha}G_{c_1,\alpha}(y))dy \quad \text{for} \quad x\in[0,s_\alpha(t)],\ \ t>0,$$

where we added a subscript α to emphasize that the solution depends on α. We rewrite also the formulas (5.61)-(5.66) with a new subscript α. Then we have

$$M_{1,\alpha}(p,y):=\frac{1}{\Gamma(1-\alpha)}\int_y^p(1-p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu,$$

$$M_{n,\alpha}(p,y):=\int_y^p M_{1,\alpha}(a,y)M_{n-1,\alpha}(p,a)da \quad \text{for} \quad n\geq 2$$

for $0\leq y\leq p$ and

$$G_{c_1,\alpha}(y)=\frac{1}{\Gamma(1-\alpha)}\int_y^{c_1}(1-c_1^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}})^{-\alpha}d\mu,$$

$$(L^n_{c_1,\alpha}G_{c_1,\alpha})(y)=\int_y^{c_1}M_{n,\alpha}(p,y)G_{c_1,\alpha}(p)dp \quad \text{for} \quad n\geq 1 \ \text{and} \ \ 0\leq y\leq c_1.$$

We would like to pass to the limit with α in the formula for u_α. Hence, at first we shall calculate the limit as $\alpha\nearrow 1$ in the formulas for $M_{n,\alpha}$, $G_{c_1,\alpha}$ and $L^n_{c_1,\alpha}G_{c_1,\alpha}$. After a substitution $q:=p^{-\frac{2}{\alpha}}\mu^{\frac{2}{\alpha}}$ we get

$$M_{1,\alpha}(p,y)=p\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}-\frac{\frac{\alpha}{2}p}{\Gamma(1-\alpha)}\int_0^{p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}}(1-q)^{-\alpha}q^{\frac{\alpha}{2}-1}dq.$$

We note that

$$\lim_{\alpha\nearrow 1}\frac{\frac{\alpha}{2}p}{\Gamma(1-\alpha)}\int_0^{p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}}(1-q)^{-\alpha}q^{\frac{\alpha}{2}-1}dq=0 \quad \text{for} \quad 0\leq y<p,$$

because for $0\leq y<p$ we have

$$\frac{\frac{\alpha}{2}p}{\Gamma(1-\alpha)}\int_0^{p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}}(1-q)^{-\alpha}q^{\frac{\alpha}{2}-1}dq\leq\frac{\frac{\alpha}{2}p}{\Gamma(1-\alpha)}(1-p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}})^{-\alpha}\int_0^{p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}}q^{\frac{\alpha}{2}-1}dq$$

$$=\frac{y}{\Gamma(1-\alpha)}(1-p^{-\frac{2}{\alpha}}y^{\frac{2}{\alpha}})^{-\alpha}\to 0 \quad \text{as} \quad \alpha\to 1.$$

Then we denote

$$M_{1,1}(p,y) := \lim_{\alpha \nearrow 1} M_{1,\alpha}(p,y) = \begin{cases} \frac{1}{2}p & \text{for} \quad 0 \le y < p, \\ 0 & \text{for} \quad 0 < y = p. \end{cases} \tag{5.86}$$

From the definition of $G_{c_1,\alpha}$ we infer that the same calculations as for $M_{1,\alpha}$ lead to

$$G_{c_1,1}(p,y) := \lim_{\alpha \nearrow 1} G_{c_1,\alpha}(p,y) = \begin{cases} \frac{1}{2}c_1 & \text{for} \quad 0 \le y < c_1, \\ 0 & \text{for} \quad y = c_1 \end{cases} \tag{5.87}$$

and

$$0 \le G_{c_1,\alpha}(y) \le c_1, \quad \text{for} \quad y \in [0,c_1], \ \ \alpha \in (0,1]. \tag{5.88}$$

From (5.67) and (5.69) we deduce that

$$0 \le M_{n,\alpha}(p,y) \le \frac{p^{2n-1}}{(n-1)!} \quad \text{for} \quad \alpha \in (0,1), \ \ 0 \le y \le p, \ \ n \ge 1. \tag{5.89}$$

The above estimates allow us to apply the Lebesgue's Dominated Convergence Theorem (LDCT) and we get

$$M_{n,1}(p,y) := \lim_{\alpha \nearrow 1} M_{n,\alpha}(p,y) = \int_y^p M_{1,1}(a,y) M_{n-1,1}(p,a)da \quad \text{for} \quad 0 \le y \le p, \ \ n \ge 2. \tag{5.90}$$

Furthermore, the estimate (5.89) gives

$$0 \le M_{n,1}(p,y) \le \frac{p^{2n-1}}{(n-1)!} \quad \text{for} \quad 0 \le y \le p, \ \ n \ge 1. \tag{5.91}$$

Applying again (5.88), (5.89) together with LDCT we obtain

$$(L^n_{c_1,1} G_{c_1,1})(y) := \lim_{\alpha \nearrow 1} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) = \int_y^{c_1} M_{n,1}(p,y) G_{c_1,1}(p)dp \quad \text{for} \quad 0 \le y \le c_1, \ \ n \ge 1. \tag{5.92}$$

Moreover, making use of (5.88), (5.89) and (5.91) we arrive at the following estimate

$$0 \le (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) \le \frac{c_1^{2n+1}}{n!} \quad \text{for} \quad 0 \le y \le c_1, \ \ n \ge 0, \ \ \alpha \in (0,1]. \tag{5.93}$$

Taking advantage of (5.92) and (5.93) we get that

$$\lim_{\alpha \nearrow 1} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) = \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y) \quad \text{for} \quad y \in [0,c_1]. \tag{5.94}$$

Recalling that $\widetilde{u}_\alpha$ was defined in 5.80, we introduce the following definition

$$u_1(x,t) := \lim_{\alpha \nearrow 1} \widetilde{u}_\alpha(x,t), \ \ t > 0, \ \ x \in [0, c_1 t^{\frac{1}{2}}]. \tag{5.95}$$

We shall characterize the above limit. If $x \in [0, c_1 t^{\frac{1}{2}})$, then we note that

$$u_1(x,t) := \lim_{\alpha \nearrow 1} \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy = \int_{xt^{-\frac{1}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y)dy, \tag{5.96}$$

where we applied (5.93), (5.94) together with LDCT. If $x = c_1 t^{\frac{1}{2}}$, then for $t \ge 1$ we have $\widetilde{u}_\alpha(c_1 t^{\frac{1}{2}}, t) = 0$ so, $u_1(c_1 t^{\frac{1}{2}}, t) = 0$. Finally, if $x = c_1 t^{\frac{1}{2}}$ and $t \in (0,1)$, then

$$\widetilde{u}_\alpha(c_1 t^{\frac{1}{2}}, t) = \lim_{\alpha \nearrow 1} u_\alpha(c_1 t^{\frac{1}{2}}, t) = \lim_{\alpha \nearrow 1} \int_{c_1 t^{\frac{1-\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy = 0,$$

where we again applied (5.93). Therefore, we deduce that

$$u_1(x,t) = \int_{xt^{-\frac{1}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y)dy, \quad \text{for } t > 0, \ \ x \in [0, c_1 t^{\frac{1}{2}}]. \tag{5.97}$$

Our next aim is to prove a uniform convergence of $\widetilde{u}_\alpha$ to u_1 on every compact subset of $\{(x,t) : 0 < t < \infty,\ \ x \in [0, c_1 t^{\frac{1}{2}}]\}$. To this end we fix $0 < t_* < t^*$ and we denote

$$Q_{t_*,t^*} = \{(x,t) :\ \ t \in [t_*, t^*],\ \ x \in [0, c_1 t^{\frac{1}{2}}]\}.$$

Then, from $u_\alpha(c_1 t^{\frac{\alpha}{2}}, t) = 0$ we deduce that $\widetilde{u}_\alpha \in C(Q_{t_*,t^*})$. We shall show that $\widetilde{u}_\alpha$ converges uniformly to u_1 on Q_{t_*,t^*}. Let us fix $\varepsilon > 0$. Without loss of generality, we may assume that $\varepsilon < 2c_1^2 e^{c_1^2}(1 - t_*^{\frac{1}{2}})$ in case of $t_* < 1$ and $\varepsilon < 2c_1^2 e^{c_1^2}(1 - t^{*-\frac{1}{2}})$ in case of $t^* > 1$. Then, from (5.93), (5.94) and LDCT we deduce that there exists $\alpha_0 \in (0,1)$ such that

$$\int_0^{c_1} \left| \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y) - \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) \right| dy \le \frac{\varepsilon}{2} \quad \text{for all} \quad \alpha \in (\alpha_0, 1). \tag{5.98}$$

To estimate $(*) := |\widetilde{u}_\alpha(x,t) - u_1(x,t)|$ for $(x,t) \in Q_{t_*,t^*}$, we have to consider three cases.

1. Case $x \in [0, c_1 t^{\frac{1}{2}}]$ and $t \le 1$. In this case we have $t_* \le 1$ and we may write

$$\begin{aligned}(*) = |u_\alpha(x,t) - u_1(x,t)| &= \left| \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy - \int_{xt^{-\frac{1}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y)dy \right| \\ &\le \int_{xt^{-\frac{\alpha}{2}}}^{xt^{-\frac{1}{2}}} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy + \int_0^{c_1} \left| \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) - \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y) \right| dy \\ &\le c_1 e^{c_1^2} x (t^{-\frac{1}{2}} - t^{-\frac{\alpha}{2}}) + \frac{\varepsilon}{2},\end{aligned}$$

where we applied (5.93) and (5.98). We define $\alpha_1 \in (0,1)$ by the equality $c_1^2 e^{c_1^2}(1 - t_*^{\frac{1-\alpha_1}{2}}) = \frac{\varepsilon}{2}$. Then, for $\alpha \in (\max\{\alpha_0, \alpha_1\}, 1)$ we have

$$\begin{aligned}(*) &\le c_1^2 e^{c_1^2} t^{\frac{1}{2}} (t^{-\frac{1}{2}} - t^{-\frac{\alpha}{2}}) + \frac{\varepsilon}{2} = c_1^2 e^{c_1^2}(1 - t^{\frac{1-\alpha}{2}}) + \frac{\varepsilon}{2} \\ &\le c_1^2 e^{c_1^2}(1 - t_*^{\frac{1-\alpha}{2}}) + \frac{\varepsilon}{2} \le c_1^2 e^{c_1^2}(1 - t_*^{\frac{1-\alpha_1}{2}}) + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

2. Case $x \in [c_1 t^{\frac{\alpha}{2}}, c_1 t^{\frac{1}{2}}]$ and $t \ge 1$. In this case we have $t^* \ge 1$ and $(*) = u_1(x,t)$. We define $\alpha_2 \in (0,1)$ by the equality $c_1^2 e^{c_1^2}(1 - t^{*\frac{\alpha_2 - 1}{2}}) = \frac{\varepsilon}{2}$. Then for $\alpha \in (\alpha_2, 1)$ we have

$$\begin{aligned}(*) &= \int_{xt^{-\frac{1}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y)dy \le c_1 e^{c_1^2}(c_1 - xt^{-\frac{1}{2}}) \le c_1^2 e^{c_1^2}(1 - t^{\frac{\alpha-1}{2}}) \le \\ &\le c_1^2 e^{c_1^2}(1 - t^{*\frac{\alpha-1}{2}}) \le c_1^2 e^{c_1^2}(1 - t^{*\frac{\alpha_2-1}{2}}) = \frac{\varepsilon}{2},\end{aligned}$$

where we applied (5.93).

3. Case $x \in [0, c_1 t^{\frac{\alpha}{2}}]$ and $t \ge 1$. In this case we have $t^* \ge 1$ and

$$\begin{aligned}(*) = |u_\alpha(x,t) - u_1(x,t)| &= \left| \int_{xt^{-\frac{\alpha}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy - \int_{xt^{-\frac{1}{2}}}^{c_1} \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y)dy \right| \\ &\le \int_{xt^{-\frac{1}{2}}}^{xt^{-\frac{\alpha}{2}}} \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y)dy + \int_0^{c_1} \left| \sum_{n=0}^{\infty} (L^n_{c_1,\alpha} G_{c_1,\alpha})(y) - \sum_{n=0}^{\infty} (L^n_{c_1,1} G_{c_1,1})(y) \right| dy \\ &\le c_1 e^{c_1^2} x (t^{-\frac{\alpha}{2}} - t^{-\frac{1}{2}}) + \frac{\varepsilon}{2},\end{aligned}$$

where we applied (5.93) and (5.98). Then, for $\alpha \in (\max\{\alpha_0, \alpha_2\}, 1)$ we have

$$\begin{aligned}(*) &\le c_1^2 e^{c_1^2} t^{\frac{\alpha}{2}} (t^{-\frac{\alpha}{2}} - t^{-\frac{1}{2}}) + \frac{\varepsilon}{2} = c_1^2 e^{c_1^2}(1 - t^{\frac{\alpha-1}{2}}) + \frac{\varepsilon}{2} \\ &\le c_1^2 e^{c_1^2}(1 - t^{*\frac{\alpha-1}{2}}) + \frac{\varepsilon}{2} \le c_1^2 e^{c_1^2}(1 - t^{*\frac{\alpha_2-1}{2}}) + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We note that in the calculations above the constant $\alpha_1 = \alpha_1(t_*)$ appears only in the case when $t_* \leq 1$ and similarly $\alpha_2 = \alpha_2(t^*)$ appears only in the case when $t^* \geq 1$. Hence in general case if $t_* > 1$ we set $\alpha_1 = 0$ and if $t^* < 1$ we set $\alpha_2 = 0$. Then we may write that for any $0 < t_* < t^*$ and any ε small enough, if $\alpha \in (\max\{\alpha_0, \alpha_1, \alpha_2\}, 1)$ then

$$|\widetilde{u}_\alpha(x,t) - u_1(x,t)| \leq \varepsilon \quad \text{for} \quad (x,t) \in Q_{t_*,t^*}$$

and as a consequence, u_1 is continuous on Q_{t_*,t^*}.

Let us calculate the formula for u_1. At first, we will show by induction that

$$M_{n,1}(p,y) = \frac{2p}{4^n(n-1)!}(p^2-y^2)^{n-1} \quad \text{for} \quad 0 \leq y < p, \quad n \in \mathbb{N}. \tag{5.99}$$

From (5.86) we see that the formula in (5.99) is fulfilled for $n = 1$. Let us fix a natural number $k \geq 2$. We assume that for any $l \in \mathbb{N}$ such that $1 \leq l \leq k$ we have

$$M_{l,1}(p,y) = \frac{2p}{4^l(l-1)!}(p^2-y^2)^{l-1}.$$

Then

$$M_{k+1,1}(p,y) = \frac{1}{2}\int_y^p aM_{k,1}(p,a)da = \frac{2p}{2\cdot 4^k(k-1)!}\int_y^p a(p^2-a^2)^{k-1}da.$$

Applying the substitution $a^2 = w$ we have

$$M_{k+1,1}(p,y) = \frac{p}{2\cdot 4^k(k-1)!}\int_{y^2}^{p^2}(p^2-w)^{k-1}dw = \frac{2p}{4^{k+1}k!}(p^2-y^2)^k$$

and we arrive at the formula (5.99) for $n = k+1$. Thus, by the principle of mathematical induction (5.99) is proven.

Let us calculate $L^n_{c_1}G_{c_1,1}$. Making use of (5.92) and (5.99) we get

$$L^n_{c_1}G_{c_1,1}(y) = \int_y^{c_1} M_{n,1}(p,y)G_{c_1,1}(p)dp = \frac{c_1}{2}\frac{2}{4^n(n-1)!}\int_y^{c_1} p(p^2-y^2)^{n-1}dp.$$

Applying the substitution $p^2 = w$ we have

$$L^n_{c_1}G_{c_1,1}(y) = \frac{c_1}{2}\frac{1}{4^n(n-1)!}\int_{y^2}^{c_1^2}(w-y^2)^{n-1}dw = \frac{c_1}{2\cdot 4^n n!}(c_1^2-y^2)^n = \frac{c_1}{2}\frac{1}{n!}\left(\left(\frac{c_1}{2}\right)^2 - \left(\frac{y}{2}\right)^2\right)^n.$$

Hence,

$$\begin{aligned} u_1(x,t) &= \int_{\frac{x}{\sqrt{t}}}^{c_1}\sum_{n=0}^{\infty} L^n_{c_1}G_{c_1,1}(y)dy = \frac{c_1}{2}\int_{\frac{x}{\sqrt{t}}}^{c_1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\left(\frac{c_1}{2}\right)^2 - \left(\frac{y}{2}\right)^2\right)^n \\ &= \frac{c_1}{2}\int_{\frac{x}{\sqrt{t}}}^{c_1} e^{(\frac{c_1}{2})^2-(\frac{y}{2})^2}dy = \frac{c_1}{2}e^{(\frac{c_1}{2})^2}\int_{\frac{x}{\sqrt{t}}}^{c_1} e^{-(\frac{y}{2})^2}dy. \end{aligned}$$

We substitute $y = 2w$ to get

$$u_1(x,t) = c_1 e^{(\frac{c_1}{2})^2}\int_{\frac{x}{2\sqrt{t}}}^{\frac{c_1}{2}} e^{-w^2}dw.$$

Setting $a = \frac{c_1}{2}$ we arrive at

$$u_1(x,t) = 2ae^{a^2}\int_{\frac{x}{2\sqrt{t}}}^{a} e^{-w^2}dw.$$

Therefore, the function u_1 together with $s_1 = c_1 t^{\frac{1}{2}}$ is a self-similar solution to the classical Stefan problem (5.81) - (5.84). For a construction to a self-similar solution to the classical Stefan problem we refer to [1, Example 1, Chapter 1.3].

To complete the proof of Theorem 5.13 we will show the convergence of (5.2) to (5.81) in the sense of distributions. Let us rewrite (5.2) in terms of $\widetilde{u}_\alpha$ and s_α:

$$D^{\alpha}_{s_\alpha^{-1}(x)} u_\alpha(x,t) - u_{\alpha,xx}(x,t) = -\frac{1}{\Gamma(1-\alpha)}(t - s_\alpha^{-1}(x))^{-\alpha} \quad \text{for} \quad t>0, \;\; x \in (0, c_1 t^{\frac{\alpha}{2}}). \tag{5.100}$$

We fix $\varphi \in C_c^\infty(Q_{t_*,t^*})$ and we multiply (5.100) by φ. Then we integrate the equation over Q_{s_α,t^*} and we arrive at

$$\int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} D^{\alpha}_{s_\alpha^{-1}(x)} u_\alpha(x,t)\varphi(x,t)dtdx - \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} u_{\alpha,xx}(x,t)\varphi(x,t)dtdx =$$

$$-\frac{1}{\Gamma(1-\alpha)} \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} (t - s_\alpha^{-1}(x))^{-\alpha}\varphi(x,t)dtdx. \tag{5.101}$$

We shall calculate the limit in all the above terms. Firstly, we note that $u_\alpha(x, s_\alpha^{-1}(x)) = 0$ and by Theorem 5.1 we have $u_\alpha(x,\cdot) \in AC[s_\alpha^{-1}(x), t^*]$. Hence, applying (2.15) and Remark 2.6 we may write

$$\int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} D^{\alpha}_{s_\alpha^{-1}(x)} u_\alpha(x,t)\varphi(x,t)dtdx$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} \int_{s_\alpha^{-1}(x)}^{t} (t-\tau)^{-\alpha} u_{\alpha,t}(x,\tau)d\tau\varphi(x,t)dtdx$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} \frac{d}{dt}\left[\int_{s_\alpha^{-1}(x)}^{t} (t-\tau)^{-\alpha} u_{\alpha}(x,\tau)d\tau\right]\varphi(x,t)dtdx.$$

If we integrate by parts and next apply the Fubini theorem we get that

$$-\frac{1}{\Gamma(1-\alpha)} \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} \int_{s_\alpha^{-1}(x)}^{t} (t-\tau)^{-\alpha} u_{\alpha}(x,\tau)d\tau\varphi_t(x,t)dtdx$$

$$= -\frac{1}{\Gamma(1-\alpha)} \int_0^{s_\alpha(t^*)} \int_{s_\alpha^{-1}(x)}^{t^*} \int_{\tau}^{t^*} (t-\tau)^{-\alpha} \varphi_t(x,t)dtu_{\alpha}(x,\tau)d\tau dx. \tag{5.102}$$

We note that

$$\frac{1}{\Gamma(1-\alpha)} \int_\tau^{t^*} (t-\tau)^{-\alpha}\varphi_t(x,t)dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_\tau^{t^*} (t-\tau)^{-\alpha}[\varphi_t(x,t) - \varphi_t(x,\tau)]dt + \varphi_t(x,\tau)\frac{(t^*-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_\tau^{t^*} (t-\tau)^{-\alpha}\int_\tau^t \varphi_{tt}(x,s)dsdt + \varphi_t(x,\tau)\frac{(t^*-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \xrightarrow[\alpha\nearrow 1]{} \varphi_t(x,\tau),$$

because $\lim_{\alpha\nearrow 1}\frac{1}{\Gamma(1-\alpha)} = 0$ and

$$\left|\int_\tau^{t^*} (t-\tau)^{-\alpha}\int_\tau^t \varphi_{tt}(x,s)dsdt\right| \le \sup_{(x,t)\in Q_{t_*,t^*}} |\varphi_{tt}(x,t)| \int_\tau^{t^*} (t-\tau)^{1-\alpha}d\tau.$$

We may write the expression (5.102) in the following way

$$-\int_0^\infty \int_0^{t^*} \chi_{[0,s_\alpha(t^*)]}(x)\chi_{[s_\alpha^{-1}(x),t^*]}(\tau)\frac{1}{\Gamma(1-\alpha)}\int_\tau^{t^*} (t-\tau)^{-\alpha}\bar{\varphi}_t(x,t)dt\bar{u}_\alpha(x,\tau)d\tau dx,$$

where $\bar{\varphi}$ and $\bar{u}_\alpha$ denote the extensions of φ and u_α by zero on $[0,\infty)\times[0,t^*]$. We recall that by (5.93) we have

$$u_\alpha(x,t) \le c_1^2 e^{c_1^2} \text{ for any } t>0 \text{ and } x \in [0, c_1 t^{\frac{\alpha}{2}}].$$

Furthermore, for α close enough to one, we get

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_\tau^{t^*}(t-\tau)^{-\alpha}\varphi_t(x,t)dt\right| \leq 2\left|\varphi_t(x,\tau)\right| \text{ for any } (x,t)\in Q_{t_*,t^*}.$$

Hence, we arrive at the following estimate

$$\left|\chi_{[0,s_\alpha(t^*)]}(x)\chi_{[s_\alpha^{-1}(x),t^*]}(\tau)\frac{1}{\Gamma(1-\alpha)}\int_\tau^{t^*}(t-\tau)^{-\alpha}\bar{\varphi}_t(x,t)dt\bar{u}_\alpha(x,\tau)\right| \leq 2\left|\varphi_t(x,\tau)\right| c_1^2 e^{c_1^2}.$$

Recalling that $\widetilde{u}_\alpha \rightrightarrows u_1$ on Q_{t_*,t^*} we may apply LDCT to obtain that

$$\lim_{\alpha\nearrow 1}\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*} D^\alpha_{s_\alpha^{-1}(x)}u_\alpha(x,t)\varphi(x,t)dtdx = -\int_0^{s_1(t^*)}\int_{s_1^{-1}(x)}^{t^*}\varphi_t(x,\tau)u_1(x,\tau)d\tau dx. \tag{5.103}$$

We apply the Fubini theorem and then integration by parts formula, together with the fact that $u_{\alpha,x}(s_\alpha(t),t) = u_\alpha(s_\alpha(t),t) = 0$ to get

$$\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*} u_{\alpha,xx}(x,t)\varphi(x,t)dtdx = \int_0^{t^*}\int_0^{s_\alpha(t)} u_{\alpha,xx}(x,t)\varphi(x,t)dxdt$$
$$= -\int_0^{t^*}\int_0^{s_\alpha(t)} u_{\alpha,x}(x,t)\varphi_x(x,t)dxdt = \int_0^{t^*}\int_0^{s_\alpha(t)} u_\alpha(x,t)\varphi_{xx}(x,t)dxdt.$$

Hence, applying again LDCT we obtain that

$$\lim_{\alpha\nearrow 1}\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*} u_{\alpha,xx}(x,t)\varphi(x,t)dtdx = \int_0^{s_1(t^*)}\int_{s_1^{-1}(x)}^{t^*} u_1(x,t)\varphi_{xx}(x,t)dtdx. \tag{5.104}$$

Finally, after integrating by parts we obtain

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*}(t-s_\alpha^{-1}(x))^{-\alpha}\varphi(x,t)dtdx$$
$$= \frac{1}{\Gamma(2-\alpha)}\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*}(t-s_\alpha^{-1}(x))^{1-\alpha}\varphi_t(x,t)dtdx.$$

We note that for every $(x,t)\in Q_{s_\alpha,t^*}$ there holds

$$(t-s_\alpha^{-1}(x))^{1-\alpha} \to 1.$$

Hence, applying again LDCT we obtain that

$$-\frac{1}{\Gamma(1-\alpha)}\int_0^{s_\alpha(t^*)}\int_{s_\alpha^{-1}(x)}^{t^*}(t-s_\alpha^{-1}(x))^{-\alpha}\varphi(x,t)dtdx$$
$$\underset{\alpha\nearrow 1}{\longrightarrow}\int_0^{s_1(t^*)}\int_{s_1^{-1}(x)}^{t^*}\varphi_t(x,t)dtdx = \int_0^{s_1(t^*)}\varphi(x,t^*)-\varphi(x,s_1^{-1}(x))dx = 0,$$

where the last equality holds, because φ vanishes on a neighborhood of the boundary Q_{t_*,t^*}. Therefore, taking into account the last equality together with (5.103) and (5.104) and regularity of u_1 we obtain that (5.100) converges to (5.81) in the distributional sense, which finishes the proof of the theorem.

□

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